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SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS (I)

By T. W. CHAUNDY (*Oxford*)

[Received 30 March 1941]

1. Ordinary equations: the first-order equation

WITHIN the last fifty years a small number of papers have appeared on the subject of singular solutions of ordinary differential equations of order exceeding one, but 'very little of a systematic character has been published', to quote one of the most recent.* To this systematic study I attempt here to make a contribution of which the scope will appear more precisely as I proceed.

Let us begin by taking stock of the well-established theory of singular solutions of the first-order equation. This theory develops most simply from Clairaut's equation

$$y-px = f(p).$$

Differentiation gives at once

$$\{x+f'(p)\}\frac{dp}{dx} = 0; \tag{1}$$

and we notice two things. The equation is solved by differentiation, and there is then algebraic factorization, so that we have the pair of alternatives: $dp/dx = 0$, which gives the general solution, and $x+f'(p) = 0$, which gives the singular solution.

Passing on to the general equation of the first order

$$\phi(x,y,p) = 0, \tag{2}$$

we can differentiate the equation $r-1$ times to give any rth derivative y_r in terms of those of lower order and so ultimately of x, y alone. Thus, choosing a convenient origin $x = a$ and associating with it an arbitrary $y = y(a)$, we define every coefficient of that Taylor series which proceeds in powers of $x-a$. This suffices to define $y(x)$ in a suitable domain and so gives a solution involving the 'arbitrary constant' $y(a)$. This is briefly the argument of Cauchy's existence theorem and provides the general or 'Cauchy's' solution containing the full number of arbitrary constants (i.e. one) corresponding to the order of the differential equation.

* J. F. Ritt (**4**), who gives fuller references than I have needed to use here.

In forming the derivatives y_r, however, we find powers of $\partial\phi/\partial p$ accumulating in the denominator, and so a first condition for the success of this process of solution is the inequality

$$\frac{\partial\phi}{\partial p} \neq 0, \tag{3}$$

which is also the condition that $\phi = 0$ should be soluble to give p in terms of x, y. Subject, then, to this condition, Cauchy's solution is valid and, moreover, includes every possible solution in the said domain. Conversely, therefore, if there can be any additional solution not covered by Cauchy's solution, it must also satisfy the equation annulling (3)

$$\frac{\partial\phi}{\partial p} = 0. \tag{4}$$

If we differentiate (2) and use (4), we see that any such singular solution must further satisfy

$$\frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0. \tag{5}$$

If there is to be a singular solution, these three equations must be consistent: they involve only the three arguments x, y, p, and so we have the well-known principle that a (first-order) differential equation written down at random has no singular solution. This is also evident algebraically, for a singular solution presupposes the choice of alternatives through factorization as at (1) in the solution of Clairaut's equation, and this will not happen 'in general'.

If on the other hand we start from the general solution $y = f(x, A)$ and, 'varying parameters', replace A by $\alpha(x)$, choosing α so that p retains its form, i.e. so that still

$$y = f(x, \alpha), \qquad p = f_x(x, \alpha),$$

we must have for the consistency of these two equations

$$\frac{\partial f}{\partial \alpha}\frac{d\alpha}{dx} = 0. \tag{6}$$

Here again are a pair of alternatives, of which $d\alpha/dx = 0$ gives $\alpha =$ constant, i.e. the general solution, and $\partial f/\partial\alpha = 0$ gives the singular solution. Here, however, we find no third equation, analogous to (5), which must also be satisfied by the singular solution, and so every general solution give rises to an associated singular solution. This leads to the well-known paradox: 'hardly any dif-

ferential equations have singular solutions; every differential equation has a general solution; every general solution leads to a singular solution'.

I should make two further points. In the case of Clairaut's equation the general theory simplifies. The general solution is of the same form as the differential equation (since it is got by putting $p =$ constant in the equation), and so $\partial\phi/\partial p$, $\partial f/\partial\alpha$ are of similar form. Moreover, (5) is now satisfied identically, and the paradox is simply resolved for this equation.

Lastly, we have to remember that conditions such as (4), (5), or (6) are necessary but not sufficient. These can introduce alternatives to the singular solution, variously interpreted (in geometrical language) as cuspidal loci, tac loci, nodal loci. Of such possibilities I wish to stand clear and I shall seek only *necessary* conditions for singular solutions.

2. Equations of order n

Some of these results extend readily to the equation of general order

$$\phi(x, y,..., y_n) = 0. \tag{7}$$

For example, Cauchy's existence theorem still shows that the general solution covers every solution of the differential equation that does not also satisfy the *separant*

$$\frac{\partial\phi}{\partial y_n} = 0. \tag{8}$$

Thus singular solutions must satisfy both (7), (8), and hence also, in analogy with (5), the equation

$$D_n\phi = 0, \tag{9}$$

where I use the notation

$$D_n \equiv \frac{\partial}{\partial x}+y_1\frac{\partial}{\partial y}+...+y_n\frac{\partial}{\partial y_{n-1}}. \tag{10}$$

If we turn to the general solution

$$y = f(x, A_1,..., A_n), \tag{11}$$

and, varying parameters, replace this by

$$y = f(x, \alpha_1,..., \alpha_n),$$

we choose the $\alpha_r(x)$ so that $y_1,..., y_n$ retain their form,

i.e. $$y_r = \frac{\partial^r}{\partial x^r}f(x, \alpha_1,..., \alpha_n) \quad (r = 1,..., n).$$

For consistency we now require, instead of the single equation (6), the system of n equations

$$\left.\begin{array}{r}\dfrac{\partial f}{\partial\alpha_1}d\alpha_1+...+\dfrac{\partial f}{\partial\alpha_n}d\alpha_n=0\\ \dfrac{\partial^2 f}{\partial\alpha_1\,\partial x}d\alpha_1+...+\dfrac{\partial^2 f}{\partial\alpha_n\,\partial x}d\alpha_n=0\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \dfrac{\partial^n f}{\partial\alpha_1\,\partial^{n-1}x}d\alpha_1+...+\dfrac{\partial^n f}{\partial\alpha_n\,\partial^{n-1}x}d\alpha_n=0\end{array}\right\}. \tag{12}$$

As before, the general solution itself is given by $d\alpha_1=0,..., d\alpha_n=0$, and so any singular solution must arise from the vanishing of the determinant of these equations, i.e. from

$$\Delta(x,\alpha_1,...,\alpha_n)\equiv\frac{\partial\left(f,\dfrac{\partial f}{\partial x},...,\dfrac{\partial^{n-1}f}{\partial x^{n-1}}\right)}{\partial(\alpha_1,...,\alpha_n)}=0. \tag{13}$$

Regarding this as an equation to give α_n in terms of $\alpha_1,..., \alpha_{n-1}$, we can substitute for α_n in (12), which then reduces to a system of $n-1$ first-order differential equations in the $n-1$ unknowns $\alpha_1,..., \alpha_{n-1}$, i.e., effectively, to a single equation of order $n-1$. This equation will have a general solution involving $n-1$ distinct constants and may, in its turn, have singular solutions, which we could similarly determine by varying parameters in the general solution. Arguing inductively in this way we may imagine that, in the most favourable circumstances, a differential equation of order n may possess, in addition to its general solution,

$$y=f(x,A_1,...,A_n),$$

a set of n distinct singular solutions of the types

$$\begin{aligned}&y=f_1(x,B_1,B_2,...,B_{n-1}),\\&y=f_2(x,C_1,...,C_{n-2}),\\&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\&y=f_{n-1}(x,H),\\&y=f_n(x),\end{aligned}$$

if H is the nth letter of the alphabet.

These will be the singular solutions of the *first, second,..., $(n-1)$th, nth types.* There is something also to be said for counting in the reverse direction and regarding these singular solutions as of *ranks*

$n-1$, $n-2$,..., 1, 0, so that 'rank' indicates the number of independent constants in the particular solution, and the indices of 'rank' and 'type' add up to the order of the original equation. In this hierarchy of solutions the general solution itself appears to differ in degree rather than in kind and may be thought of simply as the solution of type 0 and rank n.*

But this is just speculation. Bearing in mind that a differential equation hardly ever has a singular solution and equally that a differential equation can hardly ever be solved, I proceed at once to an actual example which exhibits the full range of solutions and which, like Clairaut's equation, can be solved by differentiation.

3. A particular equation

I define the generating function

$$\chi(t) \equiv \exp(y_n t + 1!\, y_{n-1} t^2 + \ldots + n!\, y t^{n+1}) \equiv \sum_{r=0}^{\infty} \phi_r t^r. \tag{14}$$

Then by differentiation in x, t respectively

$$D\chi = \chi \sum_{r=0}^{n} r!\, y_{n-r+1} t^{r+1},$$

$$t^2 \frac{\partial \chi}{\partial t} = \chi \sum_{r=1}^{n+1} r!\, y_{n-r+1} t^{r+1}.$$

Thus
$$D\chi = t^2 \frac{\partial \chi}{\partial t} + y_{n+1} t\chi - (n+1)!\, y t^{n+2} \chi,$$

and so, if $r \leqslant n+1$,

$$D\phi_r = (y_{n+1} + r - 1)\phi_{r-1}. \tag{15}$$

Again, $\partial\chi/\partial y_n = t\chi$, and so

$$\frac{\partial \phi_r}{\partial y_n} = \phi_{r-1}. \tag{16}$$

We may also notice that ϕ_n can be derived from ϕ_{n+1} by changing n into $n-1$ and y into y_1, and so y_r into y_{r+1}; and that ϕ_{n+1} is of the form $n!\, y + \mathscr{F}(y_1, \ldots, y_n)$ and so ϕ_{n-r+1} of the form

$$(n-r)!\, y_r + \mathscr{F}(y_{r+1}, \ldots, y_n).$$

The differential equation that I consider is

$$\phi_{n+1}(y, y_1, \ldots, y_n) = 0. \tag{17}$$

Differentiation gives, in virtue of (15),

$$(y_{n+1} + n)\phi_n = 0.$$

* For this reason I shall generally speak of the 'solution' of such-and-such type or rank, dropping the word 'singular'.

Thus either $y_{n+1} = -n$ or $\phi_n(y_1,\dots,y_n) = 0$. Differentiation of the second alternative similarly gives

$$(y_{n+1}+n-1)\phi_{n-1} = 0,$$

i.e. either $y_{n+1} = -(n-1)$ or else $\phi_{n-1}(y_2,\dots,y_n) = 0$. Similarly, differentiation of $\phi_{n-1} = 0$ gives $y_{n+1} = -(n-2)$ or $\phi_{n-2} = 0$, and so on. Proceeding in this way we at length accumulate a set of $n+1$ alternatives $\mathscr{A}_0, \mathscr{A}_1,\dots, \mathscr{A}_n$ of which the alternative $\mathscr{A}_r$ is

$$\left.\begin{array}{c}\phi_{n+1}(y,\dots,y_n)=0, \quad \phi_n(y_1,\dots,y_n)=0, \quad \dots, \quad \phi_{n-r+1}(y_r,\dots,y_n)=0\\ y_{n+1} = -(n-r)\end{array}\right\}. \tag{18}$$

From the way in which these equations have been found they are consistent. We can solve them if we integrate the last of them to give

$$y_{r+1} = K_1+K_2x+\dots+K_{n-r}x^{n-r-1}-\frac{x^{n-r}}{(n-r-1)!},$$

substitute for $y_{r+1},\dots, y_n$ in $\phi_{n-r+1} = 0$, which is linear in y_r, to give y_r, then substitute for $y_r,\dots, y_n$ in $\phi_{n-r+2} = 0$ to give y_{r-1}, and so on. In this way we arrive finally at an expression for y involving the $n-r$ constants K.

Thus the $n+1$ alternatives $\mathscr{A}_r$ give $n+1$ solutions of the differential equation. They are all distinct since they give distinct values of y_{n+1} (though they all satisfy $y_{n+2} = 0$). Evidently, then, the solution given by $\mathscr{A}_r$, which has $n-r$ distinct constants, is of rank $n-r$ and therefore of type r: conveniently the rank is just the value of $-y_{n+1}$ in the solution.

If (for convenience) we temporarily replace ϕ_{n+1} by the simple symbol ϕ and use (16) in (18), we see that the solution of type r satisfies the set of $r+1$ equations

$$\phi = 0, \quad \frac{\partial\phi}{\partial y_n} = 0, \quad \dots, \quad \frac{\partial^r\phi}{\partial y_n^r} = 0. \tag{19}$$

4. The general solution

The method outlined above for the general solution does not lead to the most significant form of it. I find it simplest to write down this form

$$-(n+1)!\,y = (x+A_1)^{n+1}+\dots+(x+A_n)^{n+1} \tag{20}$$

and then to verify it.

Differentiation gives

$$-(n-r+1)!\,y_r = \sum_{s=1}^{n}(x+A_s)^{n-r+1}. \tag{21}$$

Substitution in χ gives

$$\chi = \exp\left[-\sum_{s=1}^{n}\left\{(x+A_s)t+\frac{(x+A_s)^2t^2}{2}+\dots+\frac{(x+A_s)^{n+1}t^{n+1}}{n+1}\right\}\right]$$

$$= \exp\sum_{s=1}^{n}\log\{1-(x+A_s)t\}+O(t^{n+2})$$

$$= \prod_{s=1}^{n}\{1-(x+A_s)t\}+O(t^{n+2}),$$

i.e. χ reduces to a polynomial of degree n followed by terms in t^{n+2}, etc. The term in t^{n+1} is absent and so $\phi_{n+1} = 0$. Thus (20) is a solution and has the right number of constants.

More generally, had we taken for y the sum of $n-m$ terms only,

i.e. $$-(n+1)!\,y = \sum_{s=m+1}^{n}(x+A_s)^{n+1}, \tag{22}$$

we should have found that χ reduced to a polynomial of degree $n-m$ followed by terms in t^{n+2}, etc. In this case, then, $\phi_{n-m+1}, \phi_{n-m+2},\dots, \phi_{n+1}$ are all zero, and, as we easily see, $y_{n+1} = -(n-m)$. Thus (22) gives the solution of type m and rank $n-m$.

From the above form of the general solution we can get an explicit expression for the differential equation. For, if the $x+A_s$ are roots of the n-ic

$$z^n+q_1z^{n-1}+\dots+q_n = 0,$$

then, as in (21), $-(n-r+1)!\,y_r$ is the sum of the $(n-r+1)$th powers of these roots, and, by Newton's formulae,

$$\begin{aligned}
(n+1)!\,y+n!\,y_1q_1+\dots+2!\,y_{n-1}q_{n-1}+y_nq_n &= 0,\\
n!\,y_1+(n-1)!\,y_2q_1+\dots+y_nq_{n-1}-nq_n &= 0,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots &\\
y_n-q_1 &= 0.
\end{aligned}$$

Elimination of the q's gives

$$\begin{vmatrix} (n+1)!\,y & n!\,y_1 & \dots & 2!\,y_{n-1} & y_n \\ n!\,y_1 & (n-1)!\,y_2 & \dots & y_n & -n \\ \dots & \dots & \dots & \dots & \dots \\ y_n & -1 & \dots & 0 & 0 \end{vmatrix} = 0, \tag{23}$$

the determinant being of order $n+1$.

This, then, is the equation $\phi_{n+1}(y,\dots,y_n) = 0$. Precisely the determinant is $(-)^{\frac{1}{2}n(n+1)}(n+1)!\,\phi_{n+1}$. As we have seen, ϕ_n can be got

from ϕ_{n+1} by changing n into $n-1$ and y, etc., into y_1, etc. This gives $\phi_n = 0$ in the form

$$\begin{vmatrix} n!\,y_1 & (n-1)!\,y_2 & . & . & . & 2!\,y_{n-1} & y_n \\ (n-1)!\,y_2 & (n-2)!\,y_3 & . & . & . & y_n & -(n-1) \\ . & . & . & . & . & . & . \\ y_n & -1 & . & . & . & 0 & 0 \end{vmatrix} = 0, \quad (24)$$

and we can get the rest of the set

$$\frac{\partial\phi}{\partial y_n} = 0, \quad \frac{\partial^2\phi}{\partial y_n^2} = 0, \quad \frac{\partial^3\phi}{\partial y_n^3} = 0, \quad \ldots$$

in the same way.

Now (23), which is essentially the identity connecting the sums of the first, second,..., $(n+1)$th powers of n arguments, is not invalidated if some of the arguments are zero and it is thus satisfied by any of the forms (22). Similarly (24), which represents the identity connecting the sums of the first,..., nth powers of $n-1$ arguments, is satisfied by any of the forms (22) in which $m \geqslant 1$, and so on. This confirms that these forms are, in fact, the various singular solutions.

If we vary the parameters in (20), the equations corresponding to (12) are

$$\sum_{s=1}^{n} (x+\alpha_s)^n\, d\alpha_s = 0, \quad \ldots, \quad \sum_{s=1}^{n} (x+\alpha_s)\, d\alpha_s = 0, \tag{25}$$

and their determinant is

$$\begin{aligned} \Delta &= |\,(x+\alpha_s)^n \quad (x+\alpha_s)^{n-1} \quad \ldots \quad x+\alpha_s\,| \\ &= \prod_{s=1}^{n} (x+\alpha_s) \prod_{s\neq t} (\alpha_s-\alpha_t). \end{aligned}$$

Evidently it is the vanishing of the factors $x+\alpha_s$ that gives the various singular solutions; the other factors $\alpha_s-\alpha_t$ are soon explained away. For, if we put, say, $\alpha_1 = \alpha_2$ in (25), we get equations

$$2(x+\alpha_1)^r\, d\alpha_1 + \sum_{s=3}^{n} (x+\alpha_s)^r\, d\alpha_s = 0 \quad (r = 1,\ldots, n).$$

To avoid the general solution $d\alpha_s = 0$ $(s = 1,\ldots, n)$ we must have

$$||\,(x+\alpha_s)^n \quad . \quad . \quad . \quad x+\alpha_s\,|| = 0,$$

where the columns now run from $s = 2$ to $s = n$; and this gives no possibilities not covered by $\Delta = 0$.

We get the solution (22) of type m and rank $n-m$, if we equate to zero

$$x+\alpha_1, \quad ..., \quad x+\alpha_m, \quad d\alpha_{m+1}, \quad ..., \quad d\alpha_n.$$

This satisfies the equations (25). It does more, for it reduces Δ to a determinant of rank $n-m$. Here my example fails by proving, or at any rate suggesting, too much and I need to supplement it by further illustrative equations. I begin by discussing the Clairaut's equation of order n.

5. Clairaut's equation of order n

5.1. Raffy has shown* that Clairaut's equation can be readily extended to order n. We write

$$Y_r \equiv y_r - xy_{r+1} + \frac{x^2}{2!}y_{r+2} - ... + \frac{(-x)^{n-r}}{(n-r)!}y_n \quad (r = 0, 1,..., n). \tag{26}$$

Then the Clairaut's equation of order n is

$$\phi(Y, Y_1,..., Y_n) = 0. \tag{27}$$

From (26) with its form of a Taylor's series we have at once

$$DY_r = \frac{(-x)^{n-r}}{(n-r)!}y_{n+1} = \frac{\partial Y_r}{\partial y_n}y_{n+1}. \tag{28}$$

Thus

$$D\phi = y_{n+1}\sum_{r=0}^{n}\frac{(-x)^{n-r}}{(n-r)!}\frac{\partial \phi}{\partial Y_r} = y_{n+1}\frac{\partial \phi}{\partial y_n}, \tag{29}$$

and so *the condition* (9), i.e. $D_n\phi = 0$ *in the notation of* (10), *is satisfied identically for a Clairaut's equation*. This is in accordance with what we have seen to be true of the first-order Clairaut's equation.

Again $D\phi = 0$ gives either $y_{n+1} = 0$ or $\partial\phi/\partial y_n = 0$. Of these the alternative y_{n+1}, in virtue of (28), makes every Y_r constant, say $Y_r = A_r$ $(r = 0, 1,..., n+1)$. These $n+1$ constants are connected by the relation

$$\phi(A, A_1,..., A_n) = 0. \tag{30}$$

Now we can solve the $n+1$ equations (26) to get the y_r in terms of the Y_r in the form

$$y_r = Y_r + xY_{r+1} + \frac{x^2}{2!}Y_{r+2} + ... + \frac{x^{n-r}}{(n-r)!}Y_n \quad (r = 0, 1,..., n).$$

In particular

$$y = Y + xY_1 + \frac{x^2}{2!}Y_2 + ... + \frac{x^n}{n!}Y_n,$$

* M. L. Raffy (2), (3).

and so, when $y_{n+1} = 0$,

$$y = A + xA_1 + \frac{x^2}{2!}A_2 + \ldots + \frac{x^n}{n!}A_n. \tag{31}$$

Associating this with the relation (30) we have a solution of the differential equation containing n independent constants, which is therefore the general solution. Any singular solution comes from the other alternative

$$\frac{\partial\phi}{\partial y_n} = 0,$$

in accordance with the general theory.

Thus an equation of the form (27) has for its general solution *a polynomial of degree n*. Conversely, if a differential equation of order n has for its general solution a polynomial of degree n, this polynomial will have the form (31) where the $n+1$ constants must be reduced to n by some such relation as (30). This at once gives the form of ϕ and so Clairaut's equation (27). Thus the given differential equation has the same general solution as (27) and so must be identical with it. We may therefore say briefly that *the Clairaut's equation of order n is exactly the equation whose general solution is a polynomial of degree n.*

5.2. There is a rather odd relation between these Clairaut's equations (of order n) and the theory of seminvariants of a binary n-ic, which I mention here, though I cannot precisely see what bearing it has on my present argument. We connect up the y_r with the coefficients a_r of the binary n-ic by the substitutions

$$a_r \equiv r!\, y_{n-r}. \tag{32}$$

Then the operator

$$D_n \equiv \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + \ldots + y_n\frac{\partial}{\partial y_{n-1}}$$

becomes the operator

$$\frac{\partial}{\partial x} + a_0\frac{\partial}{\partial a_1} + 2a_1\frac{\partial}{\partial a_2} + \ldots + na_{n-1}\frac{\partial}{\partial a_n} \equiv \frac{\partial}{\partial x} + \Omega, \tag{33}$$

where Ω is the usual annihilator of seminvariants. We have just seen that, in Clairaut's equation, ϕ is annihilated by D_n; the transformed ϕ is accordingly annihilated by the operator (33). If x is explicitly absent from the differential equation so that it has the restricted form $\phi(y, y_1, \ldots, y_n) = 0$, then the corresponding function of the coefficients is annihilated by ϕ and is therefore a seminvariant.

More generally, if x is present, we recognize in (33) an operator analogous to $\Omega - y\partial/\partial x$, which is one of the annihilators of covariants. In fact, annihilation by (33) is linked up with invariance under the restricted substitution $x = x'+h$. If x is absent from the equation, we notice, in particular, that $y = y(x+h)$ is a solution if $y = y(x)$ is. More generally, if $c_1,\ldots, c_n$ are the roots of the n-ic equated to zero, the operator (33) is equivalent to

$$\frac{\partial}{\partial x}-\frac{\partial}{\partial c_1}-\ldots-\frac{\partial}{\partial c_n};$$

it therefore annihilates functions of the differences of $-x$, $c_1,\ldots, c_n$. For present purposes it is more relevant to note that the seminvariants (including invariants) and covariants of a binary quantic furnish a fruitful source of Clairaut's equation. A referee* has suggested consideration of the Clairaut's equation arising from the *protomorph*

$$(a_0, a_1,\ldots, a_n)(a_1, -a_0)^n.$$

This gives the Clairaut's equation

$$y-y_1\frac{y_{n-1}}{y_n}+\frac{y_2}{2!}\left(\frac{y_{n-1}}{y_n}\right)^2+\ldots+\frac{y_{n-1}}{(n-1)!}\left(\frac{-y_{n-1}}{y_n}\right)^{n-1}+\frac{y_n}{n!}\left(\frac{-y_{n-1}}{y_n}\right)^n = 0.$$

This equation has the general solution

$$y = C_1\frac{(x+a)^n}{n!}+C_2\frac{(x+a)^{n-2}}{(n-2)!}+\ldots+C_{n-1}(x+a)$$

and a singular solution of the first type, but, if my calculations are to be relied on, this singular solution is not rational if $n > 3$.

As a simpler example we may take the Clairaut's equation (often set as an examination question) given by the lineo-linear invariant I of a quartic, namely,

$$yy_4-y_1y_3+\tfrac{1}{2}y_2^2 = 0, \tag{34}$$

of which the general solution can be written according to choice as

$$y = A_0+A_1x+\frac{A_2x^2}{2!}+\frac{A_3x^3}{3!}+\frac{A_4x^4}{4!},$$

where $$A_0A_4-A_1A_3+\tfrac{1}{2}A_2^2 = 0,$$

or $$y = a_0(x-c_1)(x-c_2)(x-c_3)(x-c_4),$$

where $$\sum (c_1-c_2)^2(c_3-c_4)^2 = 0.$$

* In accordance with whose suggestions this sub-section has been remodelled.

5.3. We can reasonably extend the idea of a Clairaut's equation to one in which

$$y_{n+1} = f(x),$$

for it reduces to a proper Clairaut's equation by the substitution $y = y'+g(x)$, where $D^{n+1}g(x) = f(x)$. In this sense the equation $\phi_{n+1}(y, y_1,..., y_n) = 0$ discussed above may be regarded as a Clairaut's equation.

If we substitute
$$y' = \frac{nx^{n+1}}{(n+1)!}+y$$

and drop accents, we get the new general solution

$$(n+1)!\,y = nx^{n+1} - \sum_{s=1}^{n} (x+A_s)^{n+1}, \tag{35}$$

in which the right-hand side is evidently a polynomial of degree n. If we make the substitution in the differential equation, it becomes

$$\phi_{n+1}(Y, Y_1,..., Y_n)+O(x) = 0.$$

But, since the equation is now of Clairaut's form, the term $O(x)$ must vanish, and so the transformed differential equation, corresponding to the general solution (35), is simply

$$\phi_{n+1}(Y, Y_1,..., Y_n) = 0. \tag{36}$$

We can make similar changes in the explicit form of the equation (23). This equation (36) can be generalized as I now show.

6. A set of Clairaut's equations

6.1. Define the pair of generating functions

$$\chi(t) \equiv \exp \sum_{s=m}^{n} s!\,Y_{n-s}t^{s-m+1}, \tag{37}$$

$$\omega(t) \equiv \exp\left\{-\sum_{s=0}^{m-1} s!\,Y_{n-s}t^{m-s-1}\right\}, \tag{38}$$

where m is a positive integer (or zero) not exceeding $n+1$. If $m = 0$, the second sum is 'empty' and we do not use ω: in fact, χ is then the generating function, analogous to (14), which gives (36). Similarly, if $m = n+1$, the first sum is 'empty' and we use only ω. When $m > 0$, $\omega(t)$ contains the factor $\exp(-Y_{n-m+1})$, so that the differential equation is transcendental, and, as is not surprising, transcendental elements appear in the solution. At the end of this paper I discuss the case $m = 1$ in rather more detail as throwing some light on the general argument.

Now define $\theta(Y,\ldots,Y_{n-m})$, $\psi(Y_{n-m+1},\ldots,Y_n)$ as the coefficients of t^{n-m+1}, t^{m-1} respectively in χ, ω; and, more generally, $\theta^{(r)}$, $\psi^{(r)}$ as the coefficients of

$$t^{n-m+1} \text{ in } \left(\frac{t}{1+xt}\right)^r \chi \quad \text{and} \quad t^{m-1} \text{ in } \frac{\omega}{(x+t)^r}.$$

In (37), (38) differentiation gives

$$\begin{aligned} D\chi &= y_{n+1}\chi \sum_{s=m}^{n} (-x)^s t^{s-m+1} \\ &= (-x)^m t(1+xt)^{-1} y_{n+1}\chi + O(t^{n-m+2}), \\ D\omega &= -y_{n+1}\omega \sum_{s=0}^{m-1} (-x)^s t^{m-s-1} \\ &= -(-x)^{m-1}(1+t/x)^{-1} y_{n+1}\omega + O(t^m) \\ &= (-x)^m (x+t)^{-1} y_{n+1}\omega + O(t^m). \end{aligned}$$

Then, differentiating $t^r(1+xt)^{-r}\chi$, $(x+t)^{-r}\omega$, using these results for $D\chi$, $D\omega$, and picking out the coefficients of t^{n-m+1}, t^{m-1} respectively, we have

$$D\theta^{(r)} = \{(-x)^m y_{n+1} - r\}\theta^{(r+1)}, \qquad D\psi^{(r)} = \{(-x)^m y_{n+1} - r\}\psi^{(r+1)}.$$

Thus, writing* $\quad \phi \equiv \theta - \psi, \quad \phi^{(r)} \equiv \theta^{(r)} - \psi^{(r)},$

we have $\quad D\phi^{(r)} = \{(-x)^m y_{n+1} - r\}\phi^{(r+1)}.$

Evidently, then, the Clairaut's equation

$$\phi(Y,\ldots,Y_n) = 0 \tag{39}$$

shares the behaviour of (36) above: successive differentiation yields solutions of every type, the solution of type r being given by

$$\phi = 0, \quad \phi^{(1)} = 0, \quad \ldots, \quad \phi^{(r)} = 0, \quad (-x)^m y_{n+1} = r. \tag{40}$$

Since $\phi^{(r)} = (-x)^{-mr}\dfrac{\partial^r \phi}{\partial y_n^r}$, this shows, in conformity with (19), that the solution of type r satisfies

$$\phi = 0, \quad \frac{\partial \phi}{\partial y_n} = 0, \quad \ldots, \quad \frac{\partial^r \phi}{\partial y_n^r} = 0.$$

6.2. We notice too that values of y_{n+1} corresponding to the successive solutions are again in arithmetical progression. This is not accidental, for we can show that, if the differential equation is such that differentiation gives for the successive solutions $y_{n+1} = \xi_r(x)$,

* I use the notation $\phi^{(r)}$ instead of the more convenient ϕ_r to avoid confusion with the ϕ_r already defined in § 3.

then the ξ_r are in arithmetical progression. For, taking the equation to be $\phi = 0$ and writing $\mathscr{S}_r$ for the solution of rth type, we have on differentiation

$$D_n\phi + y_{n+1}\frac{\partial\phi}{\partial y_n} = 0,$$

in the notation of (10). Since $\mathscr{S}_1$ satisfies $y_{n+1} = \xi_1$, we get identically

$$D_n\phi = -\xi_1\frac{\partial\phi}{\partial y_n}. \tag{41}$$

For $\mathscr{S}_2$ we start from $\partial\phi/\partial y_n$ and proceed similarly, getting

$$D_n\frac{\partial\phi}{\partial y_n} = -\xi_2\frac{\partial^2\phi}{\partial y_n^2}. \tag{42}$$

Now $$\frac{\partial}{\partial y_n}D_n = D_n\frac{\partial}{\partial y_n} + \frac{\partial}{\partial y_{n-1}},$$

and so (41) gives

$$D_n\frac{\partial\phi}{\partial y_n} + \frac{\partial\phi}{\partial y_{n-1}} = -\xi_1\frac{\partial^2\phi}{\partial y_n^2},$$

i.e., from (42), $$\frac{\partial\phi}{\partial y_{n-1}} = (\xi_2 - \xi_1)\frac{\partial^2\phi}{\partial y_n^2}. \tag{43}$$

Similarly, from $\mathscr{S}_2$, $\mathscr{S}_3$ we get

$$\frac{\partial^2\phi}{\partial y_{n-1}\partial y_n} = (\xi_3 - \xi_2)\frac{\partial^3\phi}{\partial y_n^3}. \tag{44}$$

Differentiation of (43) partially in y_n then gives

$$\xi_2 - \xi_1 = \xi_3 - \xi_2,$$

and it is clear how the arithmetical progression arises.

7. The general solution

In analogy with the solution (20) of the equation first discussed, it will be found that to write in θ

$$s!\,Y_{n-s} = -\sum_{h=1}^{n-m+1}\frac{(-A_h)^{s-m+1}}{s-m+1} \quad (s = m,\ldots, n), \tag{45}$$

and in ψ

$$\left.\begin{aligned} s!\,Y_{n-s} &= \sum_{h=1}^{m-1}\frac{(-B_h)^{m-s-1}}{m-s-1} \quad (s = 0,\ldots, m-2) \\ (m-1)!\,Y_{n-m+1} &= \sum\log B - \sum\log A \end{aligned}\right\} \tag{46}$$

will give $\phi = 0$, so that the general solution can be written

$$y = \sum_{s=0}^{n-m} \sum_{h=1}^{n-m+1} \frac{(-A_h)^{n-m-s+1}x^s}{s!\,(n-s)!\,(m-n+s-1)} +$$

$$+ \sum_{s=n-m+2}^{n} \sum_{h=1}^{m-1} \frac{(-B_h)^{m-n+s-1}x^s}{s!\,(n-s)!\,(m-n+s-1)} -$$

$$-\Big(\sum_{h=1}^{n-m+1} \log A_h - \sum_{h=1}^{m-1} \log B_h\Big) \frac{x^{n-m+1}}{(m-1)!\,(n-m+1)!}. \quad (47)$$

There does not seem to be a simple expression for the solution of type r. We note first that $y_{n+1} = r(-x)^{-m}$ gives

$$s!\,DY_{n-s} = r(-x)^{s-m},$$

so that, on integration,

$$s!\,Y_{n-s} = -\frac{r(-x)^{s-m+1}}{s-m+1} + \text{constant} \quad (s \neq m-1),$$

$$(m-1)!\,Y_{n-m+1} = -r\log x + \text{constant},$$

where the constants have to be adjusted so that $\phi, \phi^{(1)},\ldots, \phi^{(r)} = 0$. It will be found that this can be done by taking for the Y_r expressions of the forms (45), (46) subject to the conditions that for some $q_0,\ldots, q_r$

$$\left.\begin{aligned} \prod (A_h t+1) &\equiv q_0 t^{n-m+1} + q_1 t^{n-m} + \ldots + q_r t^{n-m-r+1} + O(t^{n-m-r}) \\ \prod (B_h t+1) &\equiv q_r t^{m-1} + q_{r-1} t^{m-2} + \ldots + q_0 t^{m-r-1} + O(t^{m-r-2}) \end{aligned}\right\}. \quad (48)$$

From this the corresponding solution can be built up. It will be of the form

$$y = -\frac{rx^{n-m+1}\log x}{(m-1)!\,(n-m+1)!} + \mathscr{P}_{n-m+1}(x),$$

where $\mathscr{P}_{n-m+1}(x)$ is a polynomial of degree $n-m+1$ in x.*

8. Variation of parameters

To derive these singular solutions from the general solution (47) by variation of parameters appears, in general, to involve irrational substitutions, unlike those of § 4 in which we merely put $x+\alpha_r = 0$. The relations of these solutions to the determinant Δ defined in (13) can be discussed as follows.

* We can use (47), (48) to obtain an explicit expression, in determinant form, for the equations $\phi = 0$, $\phi^{(r)} = 0$, analogous to (23), (24), but the results are not particularly happy.

From their definitions it is clear that

$$(r-1)!\,\theta^{(r)} = \left(-\frac{\partial}{\partial x}\right)^{r-1}\theta^{(1)}, \qquad (r-1)!\,\psi^{(r)} = \left(-\frac{\partial}{\partial x}\right)^{r-1}\psi^{(1)},$$

so that
$$(r-1)!\,\phi^{(r)} = \left(-\frac{\partial}{\partial x}\right)^{r-1}\phi^{(1)}. \tag{49}$$

Here, of course, in these partial differentiations in x we regard x as belonging to the set of arguments $x, Y,..., Y_n$, not to the fundamental set $x, y,..., y_n$. Now from the form of χ we can isolate Y in the differential equation writing it

$$\phi \equiv Y+\psi(Y_1,..., Y_n) = 0.$$

Thus
$$y = Y+xY_1+...+\frac{x^nY_n}{n!} = \psi(Y_1,..., Y_n)-\sum_{s=1}^{n}\frac{Y_s x^s}{s!}.$$

The general solution is obtained by replacing every Y_s by a constant A_s, and, varying parameters, we replace these constants by the variable arguments α_s, getting

$$y = \psi(\alpha_1,..., \alpha_n)-\sum_{s=1}^{n}\frac{\alpha_s x^s}{s!}.$$

The $\Delta(x, \alpha_1,..., \alpha_n)$ of (13) is accordingly

$$\Delta = \begin{vmatrix} x-\dfrac{\partial\psi}{\partial\alpha_1} & \dfrac{x^2}{2!}-\dfrac{\partial\psi}{\partial\alpha_2} & . & . & . & \dfrac{x^n}{n!}-\dfrac{\partial\psi}{\partial\alpha_n} \\ 1 & x & . & . & . & \dfrac{x^{n-1}}{(n-1)!} \\ 0 & 1 & . & . & . & \dfrac{x^{n-2}}{(n-2)!} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & x \end{vmatrix}. \tag{50}$$

Expanding the determinant as

$$\Delta = \sum_{s=1}^{n}\lambda_s\left(\frac{x^s}{s!}-\frac{\partial\psi}{\partial\alpha_s}\right)$$

we find without difficulty* that

$$\lambda_s = (-)^{s-1}\frac{x^{n-s}}{(n-s)!}.$$

* For, by the properties of a Wronskian,

$$D\lambda_s = -\lambda_{s+1}, \quad \lambda_n = (-1)^{n-1}, \quad \lambda_s = 0 \quad (s < n) \quad \text{when } x = 0.$$

Thus
$$\Delta = \frac{x^n}{n!} + \sum_{s=1}^{n} (-)^s \frac{x^{n-s}}{(n-s)!} \frac{\partial \psi}{\partial \alpha_s}.$$

But
$$\frac{\partial \phi}{\partial y_n} = \frac{\partial Y}{\partial y_n} + \frac{\partial \psi}{\partial y_n} = \frac{(-x)^n}{n!} + \sum_{s=1}^{n} \frac{(-x)^{n-s}}{(n-s)!} \frac{\partial \psi}{\partial Y_s}.$$

Hence Δ is obtained from $(-)^n \partial\phi/\partial y_n$ by replacing every Y_s by α_s, and the correspondence of $\Delta = 0$, $\partial\phi/\partial y_n = 0$ is clear. More precisely, since
$$\phi^{(1)} = (-x)^{-m} \frac{\partial \phi}{\partial y_n},$$
we can write
$$\Delta = (-)^{m-n} x^m \phi^{(1)}(x, \alpha_1, \ldots, \alpha_n). \tag{51}$$

For solutions of type exceeding one we have $\phi = 0$, $\phi^{(1)} = 0$, i.e., by (49),
$$\Delta = 0, \qquad \frac{\partial \Delta}{\partial x} = 0.$$

Now Δ, regarded as a function of x, is a Wronskian, and we have
$$\frac{\partial \Delta}{\partial x} = \Delta_1,$$
where Δ_1 is the minor obtained by omitting the last row and column in Δ. Thus for solutions of type exceeding one we have
$$\Delta = 0, \qquad \Delta_1 = 0.$$

It follows from the properties of determinants that all the minors obtained by omitting either the last row or else the last column in Δ are zero. But λ_n is one of the latter minors and, as we have seen, its value is $(-1)^{n-1}$, not zero. Thus all the minors obtained by omitting the last row vanish, i.e., in the notation of (12),
$$\left\| \begin{matrix} \dfrac{\partial f}{\partial \alpha_1} & \cdot & \cdot & \cdot & \dfrac{\partial f}{\partial \alpha_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial^{n-1} f}{\partial \alpha_1 \partial x^{n-2}} & \cdot & \cdot & \cdot & \dfrac{\partial^{n-1} f}{\partial \alpha_n \partial x^{n-2}} \end{matrix} \right\| = 0. \tag{52}$$

Similarly, for solutions of type exceeding two, we have
$$\Delta, \frac{\partial \Delta}{\partial x}, \frac{\partial^2 \Delta}{\partial x^2} = 0, \quad \text{i.e. } \Delta, \Delta_1, \Delta_2 = 0,$$

where Δ_2 is the minor obtained by omitting the last row and column in Δ_1, and we deduce the equivalent conditions

$$\left\| \begin{matrix} \dfrac{\partial f}{\partial \alpha_1} & \cdot & \cdot & \cdot & \dfrac{\partial f}{\partial \alpha_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial^{n-2} f}{\partial \alpha_1\, \partial x^{n-3}} & \cdot & \cdot & \cdot & \dfrac{\partial^{n-2} f}{\partial \alpha_n\, \partial x^{n-3}} \end{matrix} \right\| = 0. \tag{53}$$

Generally for solutions of type exceeding r all the minors of Δ obtained by omitting the last r rows vanish. These conditions have been given* by Cerf for the general equation.

We may note in conclusion that

$$\Delta_r = \frac{\partial^r}{\partial x^r}\left(\frac{\partial \phi}{\partial y_n}\right) = (-)^r \frac{\partial \phi}{\partial y_{n-r}}.$$

9. The case $m = 1$

When $m = 1$, the generating function $\omega(t)$ reduces to $\exp(-Y_n)$, independent of t. In the general solution (47) the constants B are absent and the solution in question reduces to

$$y = -\sum_{s=0}^{n-1} \sum_{h=1}^{n} \frac{(-A_h)^{n-s} x^s}{s!\,(n-s)!\,(n-s)} - \frac{x^n}{n!} \sum_{h=1}^{n} \log A_h. \tag{54}$$

The form of this solution is seen more clearly if we note that

$$\frac{\partial y}{\partial A_h} = -\frac{(x-A_h)^n}{n!\,A_h}. \tag{55}$$

Thus, if, varying parameters, we replace the A_h by α_h, we have

$$\Delta = \left| -\frac{(x-\alpha_h)^n}{n!\,\alpha_h} \quad -\frac{(x-\alpha_h)^{n-1}}{(n-1)!\,\alpha_h} \quad \cdots \quad -\frac{(x-\alpha_h)}{\alpha_h} \right|,$$

representing the determinant by a typical column. This has factors $x-\alpha_h$ ($h = 1,\ldots, n$), and it will be found that the solution of type r is obtained by putting r of these factors equal to zero and replacing the other α's by constants (exactly as in the differential equation corresponding to the case $m = 0$ discussed in §§ 3, 4). The solution of type r is thus of the form

$$y = \mathscr{P}_n(x) - \frac{r x^n \log x}{n!},$$

where $\mathscr{P}_n(x)$ is a polynomial of degree n.

* G. Cerf (1).

In particular, if $n = 2$ (say), the differential equations $\phi = 0$, $\phi^{(1)} = 0$, $\phi^{(2)} = 0$ are respectively

$$2!\,Y+\tfrac{1}{2}Y_1^2-\exp(-Y_2) = 0,$$

$$Y_1-x-x^{-1}\exp(-Y_2) = 0, \qquad 1-x^{-2}\exp(-Y_2) = 0;$$

the successive solutions are

$$\begin{aligned} y &= -\tfrac{1}{2}x^2\log AA'+(A+A')x-\tfrac{1}{4}(A^2+A'^2), \\ y &= -\tfrac{1}{2}x^2\log x-\tfrac{1}{2}x^2\log A+Ax-\tfrac{1}{4}A^2, \\ y &= -x^2\log x+\tfrac{3}{2}x^2. \end{aligned}$$

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ON SUFFICIENT CONDITIONS FOR A FUNCTION INTEGRABLE IN THE CESÀRO-PERRON SENSE TO BE MONOTONIC

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1. It is well known that if a function $f(x)$ is continuous for $a \leqslant x \leqslant b$, and if the set of points x of (a, b) at which $D^+f(x) < 0$ is at most enumerable, then $f(x)$ increases for $a \leqslant x \leqslant b$. It is easy to show by means of an example that the condition that $f(x)$ be continuous cannot be omitted; it has been shown† by Zygmund that it may, however, be replaced by

$$\overline{\lim_{h\to+0}} f(x-h) \leqslant f(x) \leqslant \overline{\lim_{h\to+0}} f(x+h). \tag{1.1}$$

It is also known‡ that, if $f(x)$ is integrable in the Denjoy-Perron or C_0-P sense and if instead of $D^+f(x)$ we consider the upper right-hand C_1-derivate $C_1 D^+f(x)$, then continuity may be replaced by C_1-continuity. It is the object of this note to improve the above result and extend it to the case of a function integrable in the Cesàro-Perron sense§ of some integral order. It will be shown that

† See S. Saks (**4**) 203–4.

‡ See W. L. C. Sargent (**5**) 236–9.

§ For the definition of Cesàro-Perron integrals, see J. C. Burkill (**2**), (**3**). For convenience I give some definitions and state some results here. If $f(x)$ is integrable in the $C_{\lambda-1}$-P sense, where $\lambda \geqslant 1$, then

$$C_\lambda(f, x, x+h) = \frac{\lambda}{h^\lambda} \int_x^{x+h} (x+h-t)^{\lambda-1} f(t)\, dt,$$

$$C_\lambda\text{-}\overline{\lim_{h\to0}} f(x+h) = \overline{\lim_{h\to0}}\, C_\lambda(f, x, x+h),$$

$$C_\lambda D^+f(x) = \overline{\lim_{h\to+0}} \frac{C_\lambda(f, x, x+h) - f(x)}{h/(\lambda+1)},$$

with corresponding definitions for the other limits and derivates. Further, if $\mu > \lambda$, then $f(x)$ is integrable in the $C_{\mu-1}$-P sense, and

$$C_\lambda\text{-}\underline{\lim_{h\to0}} f(x+h) \leqslant C_\mu\text{-}\underline{\lim_{h\to0}} f(x+h) \leqslant C_\mu\text{-}\overline{\lim_{h\to0}} f(x+h) \leqslant C_\lambda\text{-}\overline{\lim_{h\to0}} f(x+h),$$

$$C_\lambda D_+f(x) \leqslant C_\mu D_+f(x) \leqslant C_\mu D^+f(x) \leqslant C_\lambda D^+f(x).$$

These results are similar to results proved by Burkill, (**3**), 543.

if $f(x)$ is integrable in the $C_{\lambda-1}$-P sense in (a,b), where λ is a positive integer, and if, for $a \leqslant x \leqslant b$,†

$$C_\lambda\text{-}\varliminf_{h\to+0} f(x-h) \leqslant f(x) \leqslant C_\lambda\text{-}\varlimsup_{h\to+0} f(x+h), \tag{1.2}$$

whilst the set of points x of (a,b) at which $C_\lambda D^+f(x) < 0$ is at most enumerable, then $f(x)$ increases for $a \leqslant x \leqslant b$.

It will be found that the extension from the case $\lambda = 1$ to the case $\lambda > 1$ can be made very easily by means of the properties of generalized absolute continuity of Cesàro-Perron integrals which have recently been established.‡

It should be noted that (1.2) is satisfied if $f(x)$ is C_μ-continuous§ for $a \leqslant x \leqslant b$, where $\mu \geqslant \lambda$. The first inequality in (1.2) is also satisfied at all points x at which $C_\lambda D^-f(x) > -\infty$, whilst the second inequality is satisfied at all points x at which $C_\lambda D^+f(x) > -\infty$. The result stated therefore includes the result†† that, if, for $a \leqslant x \leqslant b$, $f(x)$ is C_λ-continuous and $C_\lambda D_* f(x) = \min\{C_\lambda D_-f(x), C_\lambda D_+f(x)\} \geqslant 0$, then $f(b) \geqslant f(a)$, the hypothesis of C_λ-continuity not being required.

2. We begin by considering the case $\lambda = 1$ and first prove a subsidiary theorem.

THEOREM I. *If $f(x)$ is integrable in the C_0-P sense in (a,b), and if*

$$C_1\text{-}\varliminf_{h\to+0} f(x-h) \leqslant f(x) \quad (a < x \leqslant b), \tag{2.1}$$

$$C_1 D^+f(x) > 0 \qquad (a \leqslant x < b), \tag{2.2}$$

then $f(b) > f(a)$.

Since $C_1 D^+f(a) > 0$, we can find a point c such that $a < c < b$ and

$$f(c) > f(a). \tag{2.3}$$

Now $\int_c^x \{f(t)-f(c)\}\,dt$ is a continuous function of x in the closed interval $[c,b]$ and so attains an upper bound at some point ξ of the interval. Also, since $C_1 D^+f(c) > 0$, $\int_c^x \{f(t)-f(c)\}\,dt$ is greater than zero at some points of (c,b), and ξ cannot coincide with c.

† It is easy to see by means of an example that, in the corresponding inequalities (1.1), $\varlimsup_{h\to+0} f(x-h)$ cannot be replaced by $\varliminf_{h\to+0} f(x-h)$; for instance, take $f(x) = 1$ if x is irrational, $f(x) = 0$ if x is rational.

‡ W. L. C. Sargent (**6**).

§ i.e. if C_μ-$\lim_{h\to 0} f(x+h) = f(x)$.

†† This result is fundamental in the definition of the Cesàro-Perron integral by the method of major and minor functions; see J. C. Burkill (**3**), 544.

Whenever $c < \xi - h < \xi$,

$$\int_{c}^{\xi-h} \{f(t)-f(c)\}\,dt \leqslant \int_{c}^{\xi} \{f(t)-f(c)\}\,dt,$$

and hence
$$f(c) \leqslant \frac{1}{h}\int_{\xi-h}^{\xi} f(t)\,dt.$$

In view of (2.1) it follows that

$$f(c) \leqslant f(\xi). \tag{2.4}$$

Now suppose, if possible, that $\xi < b$. Then, whenever

$$\xi < \xi+h < b,$$

$$\int_{c}^{\xi+h} \{f(t)-f(c)\}\,dt \leqslant \int_{c}^{\xi} \{f(t)-f(c)\}\,dt,$$

and hence
$$\frac{1}{h}\int_{\xi}^{\xi+h} f(t)\,dt \leqslant f(c),$$

so that, by (2.4),
$$\frac{1}{h}\int_{\xi}^{\xi+h} f(t)\,dt \leqslant f(\xi).$$

It follows that $C_1 D^+ f(\xi) \leqslant 0$, which contradicts (2.2).

The point ξ must therefore coincide with b and hence, in view of (2.3) and (2.4),

$$f(b) > f(a).$$

We now prove the main theorem for the case $\lambda = 1$.

THEOREM II. *If $f(x)$ is integrable in the C_0-P sense in (a, b) and if, for $a \leqslant x \leqslant b$,†*

$$C_1\text{-}\varliminf_{h\to+0} f(x-h) \leqslant f(x) \leqslant C_1\text{-}\varlimsup_{h\to+0} f(x+h), \tag{2.5}$$

whilst the set of points x of (a, b) at which $C_1 D^+ f(x) < 0$ is at most enumerable, then $f(x)$ increases for $a \leqslant x \leqslant b$.

Take α, β such that $a \leqslant \alpha < \beta \leqslant b$ and suppose that $C_1 D^+ f(x) \geqslant 0$ except perhaps at an enumerable set of points $x_1, x_2, x_3, \ldots, x_n, \ldots$ contained in the interval $\underset{x}{E}[\alpha \leqslant x < \beta]$.

† It is to be understood that the first inequality need only be satisfied for $a < x \leqslant b$ and the second for $a \leqslant x < b$.

Let ϵ be an arbitrary positive number. Corresponding to each positive integer n, we construct a continuous increasing function $z_n(x)$ such that $z_n(\alpha) = 0$, $z_n(\beta) = 2^{-n}\epsilon$, and

$$C_1 D^+\{f(x_n)+z_n(x_n)\} \geqslant 0. \tag{2.6}$$

The construction is obtained by modifying one given by Bosanquet.†

In view of (2.5) we can find a sequence of points $\{\gamma_s\}$ contained in (x_n, β) and such that, for every positive integer s,

$$0 < \gamma_{s+1}-x_n \leqslant \tfrac{1}{2}(\gamma_s-x_n), \tag{2.7}$$

$$C_1(f, x_n, \gamma_s) > f(x_n)-2^{-(n+s)}\epsilon. \tag{2.8}$$

We now define $z_n(x)$ in such a way that $z_n(x) = 0$ for $\alpha \leqslant x \leqslant x_n$, $z_n(x) = 2^{-n}\epsilon$ for $\gamma_1 < x \leqslant \beta$, whilst

$$z_n(\gamma_s) = 2^{1-(n+s)}\epsilon \quad (s = 1, 2, 3,...), \tag{2.9}$$

and $z_n(x)$ is linear in each closed interval $[\gamma_{s+1}, \gamma_s]$.

It is easily seen that $z_n(x)$ is continuous and increases for $\alpha \leqslant x \leqslant \beta$, and that $z_n(\alpha) = 0$, $z_n(\beta) = 2^{-n}\epsilon$; it therefore only remains to show that (2.6) is satisfied.

It follows from (2.7) and (2.9) that $z_n(x)/(x-x_n)$ decreases for $x_n < x \leqslant \beta$, and hence it can be shown that

$$C_1(z_n, x_n, \gamma_s) \geqslant \tfrac{1}{2}z_n(\gamma_s) = 2^{-(n+s)}\epsilon.$$

In view of (2.8) it follows that

$$C_1(f+z_n, x_n, \gamma_s) > f(x_n) \quad (s = 1, 2, 3,...),$$

so that (2.6) is satisfied.

Now define
$$g(x) = f(x)+\sum_{n=1}^{\infty} z_n(x)+\epsilon x.$$

It can easily be seen that, whenever $\alpha \leqslant x < \beta$,

$$C_1 D^+g(x) \geqslant \epsilon > 0.$$

Moreover, it follows from (2.5) and the uniform convergence of the series of continuous functions $\sum z_n(x)$ that

$$C_1\text{-}\varliminf_{h\to+0} g(x-h) \leqslant g(x) \quad (\alpha < x \leqslant \beta).$$

It therefore follows from Theorem I that $g(\beta) > g(\alpha)$, so that

$$f(\beta) > f(\alpha)-\sum_{n=1}^{\infty}\{z_n(\beta)-z_n(\alpha)\}-\epsilon(\beta-\alpha)$$
$$= f(\alpha)-\epsilon-\epsilon(\beta-\alpha).$$

† L. S. Bosanquet (1) 163–4.

Since ϵ is arbitrary, $\qquad f(\beta) \geqslant f(\alpha).$

The function $f(x)$ therefore increases for $a \leqslant x \leqslant b$.

3. We now prove the main theorem.

THEOREM III. *If $f(x)$ is integrable in the $C_{\lambda-1}$-P sense in (a,b), where λ is a positive integer and if, for $a \leqslant x \leqslant b$,*†

$$C_\lambda\text{-}\varliminf_{h\to+0} f(x-h) \leqslant f(x) \leqslant C_\lambda\text{-}\varlimsup_{h\to+0} f(x+h), \tag{3.1}$$

whilst the set of points x of (a,b) at which $C_\lambda D^+f(x) < 0$ is at most enumerable, then $f(x)$ increases for $a \leqslant x \leqslant b$.

The result has already been proved for the case $\lambda = 1$. We therefore suppose that $\lambda > 1$.

Let K be the set of points ξ of (a,b) throughout no neighbourhood of which $f(x)$ increases (if ξ coincides with a or b, instead of neighbourhoods we consider intervals having a or b as left-hand or right-hand end point respectively). It can be shown in the usual way that K is closed and that, if (α,β) is any complementary interval of K, $f(x)$ increases for $\alpha < x < \beta$ and hence, by (3.1), for $\alpha \leqslant x \leqslant \beta$. The set K must therefore be perfect or null.

Suppose, if possible, that K is not null. Since $f(x)$ is integrable in the $C_{\lambda-1}$-P sense in (a,b), it follows from the properties of generalized absolute continuity of the $C_{\lambda-1}$-P integral‡ that K contains a portion§ k over which $f(x)$ is absolutely integrable, whilst

$$\sum_{n=1}^{\infty} \left| \int_{a_n}^{b_n} f(t)\,dt \right| < \infty,$$

where (a_1,b_1), (a_2,b_2), (a_3,b_3),..., (a_n,b_n),... are the complementary intervals of the closure $\bar{k}$ of k. Moreover, since $f(x)$ increases for $a_n \leqslant x \leqslant b_n$, $f(x)$ is absolutely integrable over each interval (a_n,b_n). It follows that $f(x)$ is absolutely integrable over (c,d), where c, d are the end points of $\bar{k}$.

† As in Theorem II, the first inequality need only be satisfied for $a < x \leqslant b$ and the second for $a \leqslant x < b$.

‡ W. L. C. Sargent (**6**) 218–19: if $f(x)$ is integrable in the C_r-P sense in (a,b), then the closed interval $[a,b]$ can be expressed as the sum of an enumerable number of closed sets over each of which $f(x)$ is absolutely integrable and $\int_a^x f(t)\,dt$ absolutely continuous; the result stated follows by an application of Baire's theorem.

§ By a portion of a closed set Q is meant the common part of Q and an open interval which contains points of Q.

The C_1-limits and C_1-derivates of $f(x)$ are therefore defined for $c \leqslant x \leqslant d$ (perhaps only one-sidedly at c and d). Since

$$C_1\text{-}\varliminf_{h\to+0} f(x-h) \leqslant C_\lambda\text{-}\varliminf_{h\to+0} f(x-h),$$

and
$$C_\lambda\text{-}\varlimsup_{h\to+0} f(x+h) \leqslant C_1\text{-}\varlimsup_{h\to+0} f(x+h),$$

it follows from (3.1) that, for $c \leqslant x \leqslant d$,

$$C_1\text{-}\varliminf_{h\to+0} f(x-h) \leqslant f(x) \leqslant C_1\text{-}\varlimsup_{h\to+0} f(x+h).$$

Moreover, since $C_1 D^+f(x) \geqslant C_\lambda D^+f(x)$, the set of points x of (c,d) at which $C_1 D^+f(x) < 0$ is at most enumerable. It therefore follows from Theorem II that $f(x)$ increases for $c \leqslant x \leqslant d$, which is impossible if c and d are the end points of the closure of a portion of K.

The set K must therefore be null, and hence $f(x)$ increases for $a \leqslant x \leqslant b$.

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ON EXPANSIONS IN EIGENFUNCTIONS (VI)

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1. In the previous paper† I obtained the expansion related to the operator

$$L \equiv \frac{d^2}{dx^2} - q(x)$$

taken over an interval one end of which is at infinity, or is a singularity of $q(x)$. In the first part of this paper I indicate briefly the corresponding method for the case where the interval extends in each direction to infinity or to a singularity of $q(x)$.† In the second part the special forms of the expansion when $q(x)$ satisfies certain special conditions are obtained.

2. We now take the interval to be $(-\infty, \infty)$, and suppose that $q(x)$ is continuous for $-\infty < x < \infty$. Let $\eta(x) = \eta(x, w)$, $\vartheta(x) = \vartheta(x, w)$ be the solutions of $(L-w)\eta = 0$ such that

$$\eta(0) = 0, \qquad \eta'(0) = -1, \qquad \vartheta(0) = 1, \qquad \vartheta'(0) = 0, \quad (2.1)$$

where dashes denote differentiations with respect to x. Then

$$W(\eta, \vartheta) = 1. \quad (2.2)$$

By the previous theory there are functions $l_1(w)$ and $l_2(w)$, regular in the upper half-plane, such that, if $w = u+iv$, $v > 0$

$$f(x, w) = \vartheta(x, w) + l_1(w)\eta(x, w)$$

is $L^2(-\infty, 0)$, and

$$g(x, w) = \vartheta(x, w) + l_2(w)\eta(x, w)$$

is $L^2(0, \infty)$. Then $\qquad W(f, g) = l_1 - l_2. \quad (2.3)$

Also, as in § 2 of (IV),

$$2v \int_0^\infty |g(x, w)|^2\, dx = i[W(g, \bar{g})]_0^\infty$$
$$= -i\{g(0)\bar{g}'(0) - g'(0)\bar{g}(0)\} = -i(l_2 - \bar{l}_2),$$

* See above, pp. 89–107.

† The result was deduced from the theory of integral equations by H. Weyl, *Göttinger Nachrichten* (1910), 442–67.

i.e.
$$\int_0^\infty |g(x,w)|^2\,dx = \mathbf{I}(l_2)/v. \tag{2.4}$$

Similarly,
$$\int_{-\infty}^0 |f(x,w)|^2\,dx = -\mathbf{I}(l_1)/v. \tag{2.5}$$

Hence $\mathbf{I}(l_1) < 0$, $\mathbf{I}(l_2) > 0$, for $v > 0$.

We define
$$\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^\infty G(x,y,w)\psi(y)\,dy,$$

where $\psi(y)$ is the arbitrary function to be expanded, and
$$G(x,y,w) = \begin{cases} \dfrac{g(x,w)f(y,w)}{l_1(w)-l_2(w)} & (y<x), \\[2ex] \dfrac{f(x,w)g(y,w)}{l_1(w)-l_2(w)} & (y>x). \end{cases}$$

It can be proved as in the previous paper that, if ψ and $L\psi$ are $L^2(-\infty,\infty)$, then
$$|\Psi_+(x,w)| \leqslant K/v \tag{2.6}$$

and
$$\int_{-\infty}^\infty |\mathbf{R}\Psi_+(x,w)|\,du \leqslant K, \tag{2.7}$$

where the K's depend on x only. To obtain these results, consider the solutions
$$f(x,w,a) = \vartheta(x,w)+l\eta(x,w), \qquad g(x,w,b) = \vartheta(x,w)+l'\eta(x,w)$$

such that
$$f(a,w,a)\cos h+f'(a,w,a)\sin h = 0,$$
$$g(b,w,b)\cos j+g'(b,w,b)\sin j = 0.$$

Let
$$\Psi_+(x,w,a,b) = -\frac{i}{\sqrt{(2\pi)}}\int_a^b G(x,y,w,a,b)\psi(y)\,dy,$$

where
$$G(x,y,w,a,b) = \begin{cases} \dfrac{g(x,w,b)f(y,w,a)}{l-l'} & (y<x), \\[2ex] \dfrac{f(x,w,a)g(y,w,b)}{l-l'} & (y>x). \end{cases}$$

The corresponding results for $\Psi_+(x,w,a,b)$ are obtained as before, and (2.6) and (2.7) follow on making $a \to -\infty$, $b \to \infty$.

It follows from (2.6) and (2.7) that

$$\phi(x,\lambda) = \lim_{v\to 0} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\lambda \mathbf{R}\Psi_+(x,w)\, du \tag{2.8}$$

exists for all real λ and x, and is of bounded variation in $-\infty < \lambda < \infty$. We also obtain as before the inequality

$$\int_{-\infty}^{\infty} \{\phi(x,\lambda)\}^2\, dx \leqslant \int_{-\infty}^{\infty} \{\psi(x)\}^2\, dx. \tag{2.9}$$

We next prove as before that

$$\psi(x) = \int_{-\infty}^{\infty} d\phi(x,\lambda). \tag{2.10}$$

Now (2.8) may be written

$$\begin{aligned}\phi(x,\lambda) &= \frac{1}{\pi} \lim_{v\to 0} \mathbf{I}\left\{\int_0^\lambda \frac{\vartheta(x,w)+l_2(w)\eta(x,w)}{l_1(w)-l_2(w)}\, du \int_{-\infty}^{x} \{\vartheta(y,w)+l_1(w)\eta(y,w)\}\psi(y)\, dy + \right.\\ &\quad \left. + \int_0^\lambda \frac{\vartheta(x,w)+l_1(w)\eta(x,w)}{l_1(w)-l_2(w)}\, du \int_x^\infty \{\vartheta(y,w)+l_2(w)\eta(y,w)\}\psi(y)\, dy\right\}.\end{aligned} \tag{2.11}$$

This gives formally

$$\begin{aligned}\phi(x,\lambda) &= \frac{1}{\pi}\int_0^\lambda \mathbf{I}\left\{\frac{1}{l_1(u)-l_2(u)}\right\}\vartheta(x,u)\, du \int_{-\infty}^{\infty} \vartheta(y,u)\psi(y)\, dy + \\ &\quad + \frac{1}{\pi}\int_0^\lambda \mathbf{I}\left\{\frac{l_1(u)}{l_1(u)-l_2(u)}\right\}\eta(x,u)\, du \int_{-\infty}^{\infty} \vartheta(y,u)\psi(y)\, dy + \\ &\quad + \frac{1}{\pi}\int_0^\lambda \mathbf{I}\left\{\frac{l_1(u)}{l_1(u)-l_2(u)}\right\}\vartheta(x,u)\, du \int_{-\infty}^{\infty} \eta(y,u)\psi(y)\, dy + \\ &\quad + \frac{1}{\pi}\int_0^\lambda \mathbf{I}\left\{\frac{l_1(u)l_2(u)}{l_1(u)-l_2(u)}\right\}\eta(x,u)\, du \int_{-\infty}^{\infty} \eta(y,u)\psi(y)\, dy.\end{aligned} \tag{2.12}$$

Hence (2.10) gives formally the expansion

$$\psi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathbf{I}\left\{\frac{1}{l_1(u)-l_2(u)}\right\}\vartheta(x,u)\,du \int_{-\infty}^{\infty} \vartheta(y,u)\psi(y)\,dy +$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty} \mathbf{I}\left\{\frac{l_1(u)}{l_1(u)-l_2(u)}\right\}\eta(x,u)\,du \int_{-\infty}^{\infty} \vartheta(y,u)\psi(y)\,dy +$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty} \mathbf{I}\left\{\frac{l_1(u)}{l_1(u)-l_2(u)}\right\}\vartheta(x,u)\,du \int_{-\infty}^{\infty} \eta(y,u)\psi(y)\,dy +$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty} \mathbf{I}\left\{\frac{l_1(u)l_2(u)}{l_1(u)-l_2(u)}\right\}\eta(x,u)\,du \int_{-\infty}^{\infty} \eta(y,u)\psi(y)\,dy. \tag{2.13}$$

In some cases (e.g. the Bessel-function expansion over $(0,\infty)$, with 0 corresponding to the above $-\infty$) $l_1(w)$ tends to a real limit $l_1(u)$ as $v \to 0$, so that

$$f(x,u) = \vartheta(x,u)+l_1(u)\eta(x,u)$$

is $L^2(-\infty,0)$. We then have formally

$$\mathbf{I}\left\{\frac{l_1(u)}{l_1(u)-l_2(u)}\right\} = l_1(u)\mathbf{I}\left\{\frac{1}{l_1(u)-l_2(u)}\right\},$$

$$\mathbf{I}\left\{\frac{l_1(u)l_2(u)}{l_1(u)-l_2(u)}\right\} = \{l_1(u)\}^2\mathbf{I}\left\{\frac{1}{l_1(u)-l_2(u)}\right\}.$$

The formal expansion is then

$$\psi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathbf{I}\left\{\frac{1}{l_1(u)-l_2(u)}\right\}f(x,u)\,du \int_{-\infty}^{\infty} f(y,u)\psi(y)\,dy. \tag{2.14}$$

A similar result is obtained if $l_2(w)$ tends to a real limit.

Since the functions occurring in the above integrals are not necessarily integrable, the procedure is so far purely formal.

3. To put the analysis into a rigorous form, we require a series of lemmas as in the previous paper.

Let the eigenvalues and eigenfunctions for the finite interval (a,b) be $w_{n,a,b}$ and $\psi_n(x,a,b)$. Then

$$\int_a^b G(x,y,w,a,b)\psi_n(y,a,b)\,dy = \frac{\psi_n(x,a,b)}{w-w_{n,a,b}}.$$

Hence $$\int_a^b |G(x,y,w,a,b)|^2\,dy = \sum_{n=1}^{\infty} \frac{\psi_n^2(x,a,b)}{(u-w_{n,a,b})^2+v^2}.$$

Hence, as in the previous paper,

$$\int_0^\lambda du \int_a^b |G(x,y,w,a,b)|^2\,dy = \sum_{n=1}^{\infty} \psi_n^2(x,a,b) \int_0^\lambda \frac{du}{(u-w_{n,a,b})^2+v^2}$$

$$= O\left\{\frac{1}{v}\sum_{n=1}^{\infty} \frac{\psi_n^2(x,a,b)}{w_{n,a,b}^2+1}\right\} = O\left\{\frac{1}{v}\int_a^b |G(x,y,i,a,b)|^2\,dy\right\} = O\left(\frac{1}{v}\right)$$

as $v \to 0$, uniformly in a and b, x and λ being fixed. Hence

$$\int_0^\lambda du \int_{-\infty}^{\infty} |G(x,y,w)|^2\,dy = O\left(\frac{1}{v}\right). \tag{3.1}$$

Taking $x = 0$, and using (2.4) and (2.5), this gives

$$\int_0^\lambda \mathbf{I}\left\{\frac{1}{l_1(w)-l_2(w)}\right\} du = O(1). \tag{3.2}$$

Arguing similarly with $G_x(x,y,w,a,b)$, we obtain

$$\int_0^\lambda \mathbf{I}\left\{\frac{l_1(w)l_2(w)}{l_1(w)-l_2(w)}\right\} du = O(1). \tag{3.3}$$

It is easily verified that

$$\left\{\mathbf{I}\left(\frac{l_1}{l_1-l_2}\right)\right\}^2 \leqslant \mathbf{I}\left(\frac{1}{l_1-l_2}\right)\mathbf{I}\left(\frac{l_1 l_2}{l_1-l_2}\right). \tag{3.4}$$

Hence also $$\int_0^\lambda \left|\mathbf{I}\left\{\frac{l_1(w)}{l_1(w)-l_2(w)}\right\}\right| du = O(1). \tag{3.5}$$

We deduce, as in the previous paper, that

$$\lim_{v\to 0} \mathbf{I} \int_0^\lambda \frac{\vartheta(x,w)}{l_1(w)-l_2(w)}\,du = \int_0^\lambda \vartheta(x,u)\,dk_1(u),$$

$$\lim_{v\to 0} \mathbf{I} \int_0^\lambda \frac{l_1(w)\vartheta(x,w)}{l_1(w)-l_2(w)}\,du = \int_0^\lambda \vartheta(x,u)\,dk_2(u),$$

$$\lim_{v\to 0} \mathbf{I} \int_0^\lambda \frac{l_1(w)l_2(w)\eta(x,w)}{l_1(w)-l_2(w)}\,du = \int_0^\lambda \eta(x,u)\,dk_3(u),$$

where $k_1(u)$ and $k_3(u)$ are non-decreasing, and $k_2(u)$ is of bounded variation. We also prove that, as $v \to 0$,

$$\int_{-\infty}^{\infty} \left\{\int_0^{\lambda} \mathbf{I}G(x,y,w)\,du\right\}^2 dy = O(1). \tag{3.6}$$

Using these results as before, we obtain the following theorem.

Let $\psi(x)$ and $L\psi(x)$ be $L^2(-\infty,\infty)$. Then

$$\psi(x) = \int_{-\infty}^{\infty} \vartheta(x,u)\,d\xi_1(u) + \eta(x,u)\,d\xi_2(u), \tag{3.7}$$

where ξ_1 and ξ_2 are of bounded variation in any finite interval; also

$$\xi_1(u) = \int_{-\infty}^{\infty} \psi(y)\chi_1(y,u)\,dy, \qquad \xi_2(u) = \int_{-\infty}^{\infty} \psi(y)\chi_2(y,u)\,dy, \tag{3.8}$$

where

$$\chi_1(y,\lambda) = \int_0^{\lambda} \vartheta(y,u)\,dk_1(u) + \eta(y,u)\,dk_2(u),$$

$$\chi_2(y,\lambda) = \int_0^{\lambda} \vartheta(y,u)\,dk_2(u) + \eta(y,u)\,dk_3(u),$$

$k_1(u)$ and $k_3(u)$ being non-decreasing, and $k_2(u)$ of bounded variation.

As before there is a corresponding theorem for all functions $\psi(x)$ of $L^2(-\infty,\infty)$, in which ordinary convergence is replaced by mean convergence.

Special Cases

4. *In the case* $(0,\infty)$, *if* $q(x) \to \infty$ *as* $x \to \infty$, *the expansion is a series.*

In the language of Hilbert and Weyl we should say that there is a point-spectrum but no continuous spectrum. The result is due to Weyl,* and we have merely to point out how it follows from our analysis.

It is proved by Weyl, by elementary arguments, that, if $q(x) \to \infty$, the function $f(x,u)$ of the previous paper has the following properties. There is a set of isolated points $u_1, u_2,\ldots$ such that $f(x,u_n)$ is $L^2(0,\infty)$, while $f(x,u) \to \pm\infty$ as $x \to \infty$ for all other values of u; and, if (u', u'') is any interval not including one of the special points, then

$$|f(x,u)| \geqslant m > 0 \quad (x > x_0,\ u' \leqslant u \leqslant u''). \tag{4.1}$$

* H. Weyl, *Math. Annalen*, 68 (1910), 220–69, Satz 9.

Now the function

$$\chi(x,\lambda) = \int_0^\lambda f(x,u)\,dk(u)$$

of the previous paper is $L^2(0,\infty)$ for every λ, since

$$\chi(x,\lambda) = \lim_{v\to 0}\int_0^\lambda \mathbf{I}g(x,w)\,du$$

and

$$\int_0^\infty \left\{\int_0^\lambda \mathbf{I}g(x,w)\,du\right\}^2 dx$$

is bounded as $v \to 0$. But by (4.1)

$$|\chi(x,u'')-\chi(x,u')| = \left|\int_{u'}^{u''} f(x,u)\,dk(u)\right| \geqslant m\{k(u'')-k(u')\}$$

for $x > x_0$. Hence $k(u'') = k(u')$. Thus $k(u)$ is constant in the intervals between the points u_n. If $k(u_n+0)-k(u_n-0) = k_n$, then

$$\chi(x,\lambda) = \sum_{0<u_n<\lambda} k_n f(x,u_n).$$

Hence

$$\xi(u) = \sum_{0<u_n<u} k_n \int_0^\infty \psi(y)f(y,u_n)\,dy,$$

$$\phi(x,\lambda) = \frac{1}{\pi}\sum_{0<u_n<\lambda} k_n f(x,u_n)\int_0^\infty \psi(y)f(y,u_n)\,dy,$$

and $$\Psi_+(x,w) = \frac{1}{i\pi\sqrt{(2\pi)}}\sum_{n=1}^\infty \frac{k_n f(x,u_n)}{u_n-w}\int_0^\infty \psi(y)f(y,u_n)\,dy.$$

Thus $\Psi_+(x,w)$ is meromorphic, and we obtain a series expansion as in (IV).

5. *In the case* $(-\infty,\infty)$, *if* $q(x)\to\infty$ *as* $x\to\infty$ *and as* $x\to-\infty$, *the expansion is a series.**

Putting $x = 0$ in (3.6), we obtain

$$\int_{-\infty}^0 \left\{\int_0^\lambda \mathbf{I}\frac{\vartheta(y,w)+l_1(w)\eta(y,w)}{l_1(w)-l_2(w)}\,du\right\}^2 dy +$$

$$+\int_0^\infty \left\{\int_0^\lambda \mathbf{I}\frac{\vartheta(y,w)+l_2(w)\eta(y,w)}{l_1(w)-l_2(w)}\,du\right\}^2 dy = O(1)$$

* Weyl's *Göttinger Nachrichten* paper, 450–1.

as $v \to 0$. It follows that $\chi_1(y, \lambda)$ is $L^2(-\infty, \infty)$ for every λ. Similarly, by considering $G_x(0, y, w)$ it can be shown that $\chi_2(y, \lambda)$ is $L^2(-\infty, \infty)$.

The elementary argument of Weyl shows that $\vartheta(x, u)$ and $\eta(x, u)$ tend exponentially to $\pm\infty$ uniformly in any finite interval (u', u'') which excludes certain isolated points. Since

$$\log\frac{\vartheta(x,u)}{\eta(x,u)} = \log\frac{\vartheta(a,u)}{\eta(a,u)} + \int_a^x \frac{dy}{\eta(y,u)\vartheta(y,u)}$$

it follows that, as $x \to \infty$, $\vartheta(x, u)/\eta(x, u)$ tends to a limit $c(u)$ uniformly in such an interval; $c(u)$ is therefore continuous and of one sign in (u', u'').

If $\psi(x) = 0$ outside the interval $(n, n+1)$, then

$$\xi_1(\lambda) = \int_0^\lambda I(u)\, dk_1(u) + J(u)\, dk_2(u),$$

where

$$I(u) = \int_n^{n+1} \psi(y)\vartheta(y,u)\, dy, \qquad J(u) = \int_n^{n+1} \psi(y)\eta(y,u)\, dy,$$

and similarly for $\xi_2(\lambda)$. Hence on multiplying (3.7) by $\psi(x)$ and integrating, we obtain the Parseval formula

$$\int_n^{n+1} \{\psi(x)\}^2\, dx = \int_{-\infty}^{\infty} I^2(u)\, dk_1(u) + 2I(u)J(u)\, dk_2(u) + J^2(u)\, dk_3(u).$$

Now $$\int_a^b \lambda^2(u)\, dk_1(u) + 2\lambda(u)\mu(u)\, dk_2(u) + \mu^2(u)\, dk_3(u) \geqslant 0$$

for any continuous $\lambda(u)$, $\mu(u)$ and any limits a, b; for the left-hand side is equal to

$$\lim_{v\to 0} \int_a^b \left[\lambda^2(u)\mathbf{I}\left\{\frac{1}{l_1(w)-l_2(w)}\right\} + \right.$$
$$\left. + 2\lambda(u)\mu(u)\mathbf{I}\left\{\frac{l_1(w)}{l_1(w)-l_2(w)}\right\} + \mu^2(u)\mathbf{I}\left\{\frac{l_1(w)l_2(w)}{l_1(w)-l_2(w)}\right\}\right] du,$$

which is not negative, by (3.4). Hence

$$\int_{u'}^{u''} I^2(u)\, dk_1(u) + 2I(u)J(u)\, dk_2(u) + J^2(u)\, dk_3(u) \leqslant \int_n^{n+1} \{\psi(x)\}^2\, dx.$$

In this inequality we can put $\psi(x) = 1$ for $n \leqslant x \leqslant n+1$, since we

can apply the inequality to a sequence $\psi_\nu(x)$ which converges in mean to this limit. Now, if $|J(u)| \geqslant M$ for $n > n_0$, $u' \leqslant u \leqslant u''$, the left-hand side is not less than

$$M^2 \int_{u'}^{u''} \left\{\frac{I^2(u)}{J^2(u)}\, dk_1(u) + 2\frac{I(u)}{J(u)}\, dk_2(u) + dk_3(u)\right\}.$$

Hence, making $n \to \infty$, we obtain

$$\int_{u'}^{u''} \{c^2(u)\, dk_1(u) + 2c(u)\, dk_2(u) + dk_3(u)\} = 0.$$

Also $$\int_{u'}^{u''} \{c(u)+\epsilon\}^2\, dk_1(u) + 2\{c(u)+\epsilon\}\, dk_2(u) + dk_3(u) \geqslant 0$$

for every ϵ. Hence

$$2\epsilon \int_{u'}^{u''} \{c(u)\, dk_1(u) + dk_2(u)\} + \epsilon^2 \int_{u'}^{u''} dk_1(u) \geqslant 0$$

for every ϵ. Hence

$$\int_{u'}^{u''} \{c(u)\, dk_1(u) + dk_2(u)\} = 0.$$

Hence

$$\chi_1(y, u'') - \chi_1(y, u') = \int_{u'}^{u''} \{\vartheta(y, u) - c(u)\eta(y, u)\}\, dk_1(u).$$

Now Weyl's elementary argument shows that, for any continuous $c(u)$, $\vartheta(y, u) - c(u)\eta(y, u)$ tends to $\pm\infty$ either as $y \to \infty$ or as $y \to -\infty$, except for certain isolated values of u. It follows as before that $k_1(u)$ is constant except at certain isolated points. Hence so are $k_2(u)$ and $k_3(u)$, and the result follows as in the previous section.

6. *In the case* $(0, \infty)$, *let* $q(x)$ *belong to* $L(0, \infty)$. *Then the expansion takes the form*

$$\psi(x) = \sum_{n=1}^{\infty} k_n f(x, u_n) \int_0^{\infty} \psi(y) f(y, u_n)\, dy + \\ + \int_{-\infty}^{0} k'(u) f(x, u)\, du \int_0^{\infty} \psi(y) f(y, u)\, dy, \tag{6.1}$$

where the u_n *are isolated positive numbers and* $k'(u)$ *is integrable over any interval* $(-u_0, 0)$.

In the language of Weyl, there is a point-spectrum in $(-\infty, 0)$ and a continuous spectrum in $(0, \infty)$, the spectrum in Weyl corresponding to the negative of the set of eigenvalues in our notation. Weyl proves a similar theorem,* but with heavier restrictions on $q(x)$.

Let us write $w = -s^2$ and $f(x, w) = f_s(x) = f(x)$, so that $f_s(x)$ is the solution of

$$\frac{d^2f}{dx^2}+\{s^2-q(x)\}f = 0 \quad (0 \leqslant x < \infty)$$

such that $f_s(0) = \sin h$, $f_s'(0) = -\cos h$. Then

$$\int_0^x \sin s(x-y)\,q(y)f(y)\,dy = \int_0^x \sin s(x-y)\,\{f''(y)+s^2f(y)\}\,dy.$$

Integrating by parts twice,

$$\int_0^x \sin s(x-y)f''(y)\,dy = \sin sx\cos h+sf(x)-s\cos sx\sin h$$
$$-s^2\int_0^x \sin s(x-y)f(y)\,dy.$$

Hence

$$f(x) = \cos sx\sin h-\frac{\sin sx}{s}\cos h+\frac{1}{s}\int_0^x \sin s(x-y)\,q(y)f(y)\,dy. \quad (6.2)$$

For a fixed positive s, choose a so large that

$$\int_a^\infty |q(y)|\,dy \leqslant \tfrac{1}{2}s.$$

If $f(x)$ is unbounded, there are arbitrarily large values of x such that $|f(y)| \leqslant f(x)$ for $0 \leqslant y \leqslant x$. For such x

$$|f(x)| \leqslant 1+\frac{1}{s}+\frac{1}{s}\int_0^a |q(y)f(y)|\,dy+\frac{|f(x)|}{s}\int_a^x |q(y)|\,dy$$
$$\leqslant 1+\frac{1}{s}+\frac{1}{s}\int_0^a |q(y)f(y)|\,dy+\tfrac{1}{2}|f(x)|,$$
$$|f(x)| \leqslant 2+\frac{2}{s}+\frac{2}{s}\int_0^a |q(y)f(y)|\,dy.$$

Since this is false if $|f(x)|$ is large enough, $f(x)$ is bounded. It is also

* § 21 of the *Annalen* paper.

easily seen that it is uniformly bounded for $0 < s_0 \leqslant s \leqslant s_1$. Hence (6.2) gives

$$f(x) = \cos sx \sin h - \frac{\sin sx}{s} \cos h + \frac{1}{s} \int_0^\infty \sin s(x-y)\, q(y) f(y)\, dy + O\left\{\int_x^\infty |q(y)|\, dy\right\}$$

$$= \alpha(s)\cos sx + \beta(s)\sin sx + o(1),$$

where

$$\alpha(s) = \sin h - \frac{1}{s} \int_0^\infty \sin sy\, q(y) f_s(y)\, dy,$$

$$\beta(s) = -\frac{\cos h}{s} + \frac{1}{s} \int_0^\infty \cos sy\, q(y) f_s(y)\, dy.$$

Since the integrals converge uniformly, $\alpha(s)$ and $\beta(s)$ are continuous functions of s. Similarly, if $F_s(x)$ is the solution such that

$$F_s(0) = \cos h, \qquad F'_s(0) = \sin h,$$

then

$$F_s(x) = \gamma(s)\cos sx + \delta(s)\sin sx + o(1),$$

where

$$\gamma(s) = \cos h - \frac{1}{s} \int_0^\infty \sin sy\, q(y) F_s(y)\, dy,$$

$$\delta(s) = \frac{\sin h}{s} + \frac{1}{s} \int_0^\infty \cos sy\, q(y) F_s(y)\, dy.$$

Again

$$\int_0^\infty \sin sy\, q(y) f_s(y)\, dy = \lim_{x\to\infty} \int_0^x \sin sy\, \{f''_s(y) + s^2 f_s(y)\}\, dy$$

$$= \lim_{x\to\infty} \{\sin sx\, f'_s(x) - s \cos sx\, f_s(x) + s \sin h\}.$$

Hence

$$\alpha(s) = \lim_{x\to\infty} \left\{\cos sx\, f_s(x) - \frac{\sin sx}{s} f'_s(x)\right\}.$$

Similarly,

$$\beta(s) = \lim_{x\to\infty} \left\{\sin sx\, f_s(x) + \frac{\cos sx}{s} f'_s(x)\right\},$$

$$\gamma(s) = \lim_{x\to\infty} \left\{\cos sx\, F_s(x) - \frac{\sin sx}{s} F'_s(x)\right\},$$

$$\delta(s) = \lim_{x\to\infty} \left\{\sin sx\, F_s(x) + \frac{\cos sx}{s} F'_s(x)\right\}.$$

Since $W(f, F) = 1$, it follows that

$$\alpha(s)\delta(s)-\beta(s)\gamma(s) = 1/s. \tag{6.3}$$

Hence $\alpha(s)$ and $\beta(s)$ cannot both vanish for any positive s.

Now consider complex values of s. Let $s = \sigma-it$ ($t > 0$), and let $\phi(x) = f(x)e^{-xt}$. Then

$$e^{xt}\phi(x) = \cos sx \sin h - \frac{\sin sx}{s}\cos h + \frac{1}{s}\int_0^x \sin s(x-y)\, q(y)\phi(y)e^{yt}\, dy,$$

$$|\phi(x)| \leqslant 1+\frac{1}{|s|}+\frac{1}{|s|}\int_0^x |q(y)\phi(y)|\, dy.$$

It follows as before that $\phi(x)$ is bounded, i.e. that, as $x \to \infty$,

$$f_s(x) = O(e^{xt}),$$

and this is uniform in any finite s-region excluding $s = 0$. Hence

$$f_s(x) = \tfrac{1}{2}\sin h\, e^{isx} - \frac{\cos h}{2is}e^{isx} + O(e^{-xt}) +$$

$$+\frac{1}{2is}\int_0^x e^{is(x-y)}q(y)f_s(y)\, dy + O\left\{\int_0^x e^{-t(x-y)}|q(y)||f_s(y)|\, dy\right\}.$$

The last term is clearly $o(e^{xt})$. Also

$$\int_x^\infty e^{is(x-y)}q(y)f_s(y)\, dy = O\left\{e^{xt}\int_x^\infty |q(y)|\, dy\right\} = o(e^{xt}).$$

Hence

$$f_s(x) = \tfrac{1}{2}e^{isx}\left\{\sin h - \frac{\cos h}{is} + \frac{1}{is}\int_0^\infty e^{-isy}q(y)f_s(y)\, dy\right\} + o(e^{xt}).$$

Similarly,

$$F_s(x) = \tfrac{1}{2}e^{isx}\left\{\cos h + \frac{\sin h}{is} + \frac{1}{is}\int_0^\infty e^{-isy}q(y)F_s(y)\, dy\right\} + o(e^{xt}).$$

Now it is shown in the previous paper that there is a function $g(x) = F(x)+l_1 f(x)$ which is $L^2(0,\infty)$. Hence

$$l_1 = -\frac{\cos h + \dfrac{\sin h}{is} + \dfrac{1}{is}\displaystyle\int_0^\infty e^{-isy}q(y)F_s(y)\, dy}{\sin h - \dfrac{\cos h}{is} + \dfrac{1}{is}\displaystyle\int_0^\infty e^{-isy}q(y)f_s(y)\, dy}. \tag{6.4}$$

As s tends to a real limit, the numerator and denominator tend to $\gamma(s)-i\delta(s)$ and $\alpha(s)-i\beta(s)$ respectively. Since the latter is not zero,

$$\lim l_1 = -\frac{\gamma(s)-i\delta(s)}{\alpha(s)-i\beta(s)},$$

and the imaginary part of this is

$$\frac{\alpha(s)\delta(s)-\beta(s)\gamma(s)}{\alpha^2(s)+\beta^2(s)} = \frac{1}{s\{\alpha^2(s)+\beta^2(s)\}}.$$

Hence, in the notation of the general theorem of the previous paper, for $u<0$

$$k(u) = -\int\limits_u^0 \frac{du'}{\surd(-u')\{\alpha^2(\surd(-u'))+\beta^2(\surd(-u'))\}} = -2\int\limits_0^{\surd(-u)} \frac{ds}{\alpha^2(s)+\beta^2(s)}. \tag{6.5}$$

This gives the integral term in (6.1).

Next let $s \to -it$, where t is real and positive. The integrals in (6.4) converge uniformly, and hence the numerator and denominator in (6.4) tend to the real limits

$$\cosh h + \frac{\sinh h}{t} + \frac{1}{t}\int\limits_0^\infty e^{-ty} q(y) F_{-it}(y)\,dy,$$

$$\sinh h - \frac{\cosh h}{t} + \frac{1}{t}\int\limits_0^\infty e^{-ty} q(y) f_{-it}(y)\,dy.$$

Hence $\lim \mathbf{I}(l_1) = 0$ except possibly at the zeros of the denominator of (6.4). Since the denominator is an analytic function of s, regular for $\mathbf{I}(s) > 0$, the zeros are isolated points. At these points $k(u)$ may have discontinuities, of magnitude k_n say, and we obtain the series term in (6.1).

Under more special conditions the point-spectrum is finite or null.* A simple example is the Bessel-function case

$$q(x) = (\tfrac{1}{4}-\nu^2)/x^2 \quad (0 < a \leqslant x < \infty).$$

There is a similar theorem for the interval $(-\infty, \infty)$, when $q(x)$ belongs to $L(-\infty, \infty)$. The proof is a little more complicated, but proceeds on the same general lines.

* Cf. Weyl's *Annalen* paper, §§18–21.

CERTAIN q-IDENTITIES

By F. H. JACKSON (*Eastbourne*)

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1. The following q-identities may be of some interest and are, I believe, new.

$$\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}(a)_r(b)_r(c)_r(d)_r(a)_{2n-r}(b)_{2n-r}(c)_{2n-r}(d)_{2n-r}q^{(n,r)}$$
$$=(-1)^n(a)_n(b)_n(c)_n(d)_n(q^{n+1})_n(abq^n)_n(bcq^n)_n(caq^n)_n\,d^nq^{\frac{1}{2}n(3n-1)} \quad (1)$$

when $abcd=q^{1-3n}$. Here

$$(a)_n\equiv(a-1)(aq-1)...(aq^{n-1}-1),\qquad (n,r)\equiv\tfrac{1}{2}r(6n-3r+1),$$

and $\binom{2n}{r}$ denotes the q-analogue of the binomial coefficient

$$(2n)!/(r)!\,(2n-r)!$$

where $(n)!$ means $(q)_n$, etc.

$$\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}(a)_r(b)_r(a)_{2n-r}(b)_{2n-r}q^{(n,r)}=(a)_n(b)_n(q^{n+1})_n(abq^n)_n, \quad (2)$$

where now there is no restriction on a, b and

$$(n,r)\equiv\tfrac{1}{2}r(2n-r+1).$$

$$(a)_{2n}(b)_{2n}+\sum_{r=1}^{2n}\binom{2n}{r}(a)_{2n-r}(b)_{2n-r}(a)_r(b)_r\frac{(c)_{2n}}{(c)_r(c)_{2n-r}}q^{\frac{1}{2}r}$$
$$=(a)_n(b)_n(q^{n+1})_n(abq^n)_n(aq^{\frac{1}{2}})_n(bq^{\frac{1}{2}})_n/(c)_n(q^{\frac{1}{2}})_n, \quad (3)$$

where $$c=abq^{\frac{1}{2}}.$$

A special case of the second identity, with an ***unusual solitary factor***, is

$$1+\sum_{r=1}^{n}(-1)^rq^{(n,r)}=(1-q^n)(1-q^{n-1})...(1-q^{\frac{1}{2}n+1}). \quad (4)$$

where n is even, and

$$(n,r)\equiv\tfrac{1}{2}r(n-r+1).$$

Dr. W. N. Bailey kindly drew my attention to this curious identity for an *apparently simple* product.

2. The first two identities are obtained by a rather curious transformation of a 'q-theorem' given by the present writer,* which is of

* **(5)** 595.

interest in connexion with certain theorems due to Dougall, Rogers, and Ramanujan.* The q-theorem which is a q-basic generalization of Dougall's identity is of interest in connexion with proofs of the Rogers-Ramanujan series and equivalent products

$$\sum \frac{q^{n^2}}{(1-q^n)!} = \prod_0^\infty \frac{1}{(5r+1)(5r+4)},$$

$$\sum \frac{q^{n^2+n}}{(1-q^n)!} = \prod_0^\infty \frac{1}{(5r+2)(5r+3)},$$

and has been discussed in this aspect by Hardy.†

The second identity of this paper was stated without proof by the present writer some years ago,‡ and, as no proof has yet appeared, a very brief proof is given in this paper.

The transformation by which the first two identities are obtained depends on the fact that, under certain conditions, a set of factors in the general term of a q-hypergeometric series, for example

$$\binom{2n}{r}(q^{n-r}+1)q^{(n,r)},$$

may be split into a sum of factors such as

$$\binom{2n}{r}q^{(n,r)}+\binom{2n}{2n-r}q^{(n,2n-r)},$$

and then a factor which appears as a constant factor in every term of the series can be removed from the series and transferred to the product side of the identity, where it finds another odd factor $(1-q^n)$ waiting to combine with it. In the result the series is transformed from one of $n+1$ terms into a new *well-formed* q-hypergeometric series of $2n+1$ terms. The new series and its equivalent product possess a symmetry lacking in the original product and series. I emphasize the term *well-formed*, because, although any hypergeometric series can be divided by splitting a numerator term, it is only in special cases that the resulting longer series will be *well-formed*, that is, a series in which the factors of the successive terms and *especially* the indices of the solitary factors of q-series are formed according to definite laws. I give this explanation here to avoid the printing of long and tedious algebra connected with the transformation.

* (**5**) 595. † (**3**) 109–12. ‡ (**4**) 21, notes.

The third more complex identity is obtained from a q-generalization of a theorem relating to products of

$$ {}_2\Phi_1(\alpha, \beta; \gamma; x),\ {}_2\Phi_1(\alpha, \beta; \gamma; qx) $$

due to Watson and the present writer* using different methods.

Dr. Bailey, to whom I am indebted for kind criticism, has pointed out that the *curious* transformation by which the identities are obtained should have further interesting applications.

In any case, I think their simplicity of form and symmetry give them a character of their own in relation to what may be termed binomial and Vandermonde forms.

3. Proof of (1). The following identity is a q-generalization of Dougall's identity:†

$$ \frac{[x+y+c+1]_n[y+z+c+1]_n[x+z+c+1]_n[c+1]_n}{[x+c+1]_n[y+c+1]_n[z+c+1]_n[x+y+z+c-1]_n} = $$

$$ \sum_{r=0}^{n} \frac{(-1)^r[c+2r][c]_r[-x]_r[-y]_r[-z]_r[-n]_r[x+y+z+2c+n+1]_r q^r}{[c][r]![x+c+1]_r[y+c+1]_r[z+c+1]_r[n+c+1]_r[-x-y-z-c-n]_r}, \tag{5} $$

in which $\qquad [x]_r \equiv (q^x-1)(q^{x+1}-1)\ldots(q^{x+r-1}-1).$

This is transformed by making

$$ c \to -2n, \qquad x+y+z+2c+n+1 = -w. $$

Also all factors in which positive x, y, z appear are changed into new factors containing negative x, y, z. Thus $[x+c+1]_r$ is changed into

$$ [-x+2n-r]_r\, q^{\frac{1}{2}r}(2x-4n+r+1), $$

and

$$ \frac{[c+2r]}{[c]}\,\frac{[-n]_r}{[n+c+1]_r} $$

becomes $(q^n+q^r)/(q^n+1)$, except when $r=n$, when it becomes $q^n/(q^n+1)$. In consequence of these changes the solitary factor becomes $q^{\frac{1}{2}r(6n-3r-1)}$. The identity now is

$$ \frac{[-x-y+n]_n[-y-z+n]_n[-x-z+n]_n}{[-x+n]_n[-y+n]_n[-z+n]_n[-w+n]_n}(q^n-1)(q^{n+1}-1)\ldots(q^{2n-1}-1) $$

$$ = \sum_{r=0}^{n} \binom{2n}{r} \frac{(-1)^r[-x]_r[-y]_r[-z]_r[-w]_r}{[-x+2n-r]_r[-y+2n-r]_r[-z+2n-r]_r[-w+2n-r]_r} \times $$

$$ \times q^{(r,n)} \frac{(q^n+q^r)}{(q^n+1)}, $$

* (6) 10, 11. † (2) equation (6).

in which the factor $(q^n+q^r)/(q^n+1)$ is halved when $r = n$ and

$$(r,n) = \tfrac{1}{2}r(6n-r-1).$$

The transformation involved the summation of seventeen arithmetical progressions, which resulted in the change of the index of the solitary factor from r to $\frac{1}{2}r(6n-3r-1)$.

On removing the factor (q^n+1) from series to product, and multiplying both product and series by

$$[-x]_{2n}[-y]_{2n}[-z]_{2n}[-w]_{2n},$$

also noting that such pairs as

$$\frac{[-x]_{2n}}{[-x+2n-r]_r}$$

reduce to $$[-x]_{2n-r},$$

and finally changing the notation into a simpler form by giving to

$$q^{-x},\ q^{-y},\ q^{-z},\ q^{-w}$$

the values $$a,\ b,\ c,\ d,$$

we have the identity

$$(abq^n)_n(bcq^n)_n(acq^n)_n(q^{n+1})_n(a)_n(b)_n(c)_n(d)_n$$
$$\equiv \sum_{r=0}^{n} (-1)^r \binom{2n}{r}(a)_r(b)_r(c)_r(d)_r(a)_{2n-r}(b)_{2n-r}(c)_{2n-r}(d)_{2n-r}(q^r+q^n)q^{(r,n)}.$$

It remains to convert this into a *well-formed* series of $2n+1$ terms. Since the index of the solitary factor is

$$\tfrac{1}{2}r(6n-3r-1),$$

we find on giving r the successive values $0, 1, 2,\ldots, n$ that

$$q^{r+(r,n)},\quad \ldots,\quad q^{n+(r,n)}$$

give the sequence of factors with indices

$$\{0,\ 3n-1,\ 6n-5,\ 9n-12,\ldots,\ 7n-7,\ 4n-2,\ n\}$$

formed according to the norm $\frac{1}{2}r(6n-3r+1)$ in which r ranges from 0 to $2n$. The series thus takes its final form of $2n+1$ terms

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r}(a)_r(b)_r(c)_r(d)_r(a)_{2n-r}(b)_{2n-r}(c)_{2n-r}(d)_{2n-r}\, q^{(n,r)}$$
$$\equiv (-1)^n(a)_n(b)_n(c)_n(d)_n(q^{n+1})_n(abq^n)_n(bcq^n)_n(acq^n)_n\, d^n q^{\frac{1}{2}n(3n-1)},$$

in which $abcd = q^{1-3n}$ and $(n,r) = \frac{1}{2}r(6n-3r+1)$.

4. Proof of (2). This theorem was stated (without proof) by the present writer many years ago,* and can be derived from the q-identity (5). If we make the parameter z infinite and also make $c = -2n$, certain factors in the series combine; thus

$$\frac{[c+2r]}{[c]}\frac{[-n]_r}{[n+c+1]_r} \equiv \frac{q^r+q^n}{q^n+1},$$

except when $r = n$, and we obtain

$$\frac{[x+y+c+1]_n[c+1]_n}{[x+c+1]_n[y+c+1]_n} = \sum_{r=0}^{n}(-1)^r\binom{2n}{r}\frac{[-x]_r[-y]_r}{[x+c+1]_r[y+c+1]_r}\frac{(q^r+q^n)}{(q^n+1)}q^{(n,r)},$$

where $$(n,r) = \tfrac{1}{2}r(r+1-4n).$$

As in the case of Theorem (1), it is necessary to change factors so that all the parameters x, y appear with the minus sign. It is not necessary to give the detailed algebra. As before, the series can be rearranged into a well-formed series of $2n+1$ terms, in which the index of the solitary factor is $\frac{1}{2}r(2n-r+1)$. Finally, replacing q^{-x}, q^{-y} by a, b, we have the identity

$$(a)_n(b)_n(q^{n+1})_n(abq^n)_n = \sum_{0}^{2n}(-1)^r\binom{2n}{r}(a)_r(b)_r(a)_{2n-r}(b)_{2n-r}q^{\frac{1}{2}r(2n-r+1)},$$

without restriction on a, b.

5. Proof of (3). In a recent number of this Journal I gave a theorem† concerned with products of q-hypergeometric series

$${}_2\Phi_1(x)\times{}_2\Phi_1(qx)$$

derived from a more general theorem about solutions of q^θ equations of the form

$$\{q^{4\theta}+f_1(x)q^{2\theta}+f_2(x)\}y = 0,$$

which generalizes in q-form results about ordinary hypergeometric series due to Appell‡ and Goursat. A simple case of the q-theorem is that, when $\gamma = \alpha+\beta+\frac{1}{2}$,

$${}_2\Phi_1\{[(2\alpha)],[(2\beta)];[(2\gamma)];x\}\times{}_2\Phi_1\{[(2\alpha)],[(2\beta)];[(2\gamma)];qx\}$$
$$= {}_3\Phi_1\{[2\alpha],[2\beta],[(2\gamma-1)];[(2\gamma)],[2\gamma-1];x\}.$$

The doubled brackets [( )] signify factors advancing by q^2; and in the series for the ${}_2\Phi_1$ the factors in the denominator (analogous to

* (4) 21, notes. † (6) 10, 11. ‡ (1).

factorials) are q^2-1, q^4-1,... on the left and $q-1$, q^2-1,... on the right.

By equating coefficients of x^{2n} on multiplying in the product, an identity is obtained, which by elementary transformations gives, on changing q^2, q^α, q^β, q^γ to q, a, b, c respectively,

$$(a)_{2n}(b)_{2n}+\sum_{r=1}^{2n} q^{\frac{1}{2}r}\binom{2n}{r}(a)_r(b)_r(a)_{2n-r}(b)_{2n-r}\frac{(c)_{2n}}{(c)_r(c)_{2n-r}}$$
$$= (a)_n(b)_n(q^{n+1})_n(abq^n)_n\frac{(aq^{\frac{1}{2}})_n(bq^{\frac{1}{2}})_n}{(q^{\frac{1}{2}})_n(abq^{\frac{1}{2}})_n}.$$

On comparing this with the identity (1) we see that a relation is established between the series in the identities (2), (3) stated in the introduction. Since a and b, in the second identity, are without attached conditions, we may write briefly

$$\text{series (3)} = \text{series (2)}\times\frac{(aq^{\frac{1}{2}})_n(bq^{\frac{1}{2}})_n}{(q^{\frac{1}{2}})_n(abq^{\frac{1}{2}})_n}.$$

The results of this paper may indicate that the Dougall identity and its q-generalization could be derived through theorems relating to products of hypergeometric series, just as the series and product due to Saalschütz* are derivable.

* (7) 279 (1).

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A NOTE ON CERTAIN q-IDENTITIES

By W. N. BAILEY (*Manchester*)

[Received 3 October 1941]

THE q-identities given by Dr. F. H. Jackson* are more elegant in the form in which he gives them than when they are written in hypergeometric form, but it seems worth while to point out how surprising they are when considered as basic hypergeometric series.

Jackson's basic analogue† of Dougal's theorem for ordinary hypergeometric series gives the sum of the series

$$ {}_8\Phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} q\right] \tag{1} $$

when $a^2q = bcdef$, and f is of the form q^{-N} where N is a positive integer. Substituting for e and making $N \to \infty$, we get the sum of the infinite well-poised series

$$ {}_6\Phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d \end{matrix} aq/bcd\right], $$

where there is now no special relation connecting the parameters. To get rid of the special numerator parameters $q\sqrt{a}$, $-q\sqrt{a}$, we can take $c = \sqrt{a}$, $d = -\sqrt{a}$, and so find the sum of the series

$$ {}_2\Phi_1\left[\begin{matrix} a, & b; \\ & aq/b \end{matrix} -q/b\right] $$

corresponding to Kümmer's theorem for ordinary hypergeometric series. If we take only $d = \sqrt{a}$, we get the sum of

$$ {}_4\Phi_3\left[\begin{matrix} a, & -q\sqrt{a}, & b, & c; \\ & -\sqrt{a}, & aq/b, & aq/c \end{matrix} q\sqrt{a}/bc\right], $$

which can be written as

$$ \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r}{(q)_r(aq/b)_r(aq/c)_r} \frac{1+q^r\sqrt{a}}{1+\sqrt{a}} \left(\frac{q\sqrt{a}}{bc}\right)^r, $$

* See the preceding paper.

† See, for example, my tract *Generalized Hypergeometric Series* (Cambridge, 1935), § 8.3.

a particular case of which was pointed out many years ago by Jackson.* This is a basic generalization of Dixon's theorem for the sum of the series

$$ {}_3F_2\left[\begin{matrix} a, & b, & c; \\ & 1+a-b, & 1+a-c \end{matrix} 1\right], $$

but one would naturally prefer to obtain the sum of a series without the factor $1+q^r\sqrt{a}$. That is, we should prefer the sum of the series

$$ {}_3\Phi_2\left[\begin{matrix} a, & b, & c; \\ & aq/b, & aq/c \end{matrix} x\right] $$

for some particular value of x which $\to 1$ as $q \to 1$. For some years I have thought that no such generalization could exist, but Jackson's second identity shows that *there is such a generalization in the case when the series terminates because of the form of the parameter a and the number of terms is odd.* Thus his Theorem 2 gives the sum of the series

$$ {}_3\Phi_2\left[\begin{matrix} a, & b, & c; \\ & aq/b, & aq/c \end{matrix} q^2\sqrt{a}/bc\right] \tag{2} $$

when a is of the form q^{-2N}.

More generally, his Theorem 1 gives the sum of

$$ {}_5\Phi_4\left[\begin{matrix} a, & b, & c, & d, & e; \\ & aq/b, & aq/c, & aq/d, & aq/e \end{matrix} q^2\right] \tag{3} $$

where $bcde = qa^{\frac{3}{2}}$, and a has the above special form, and his Theorem 3 gives the sum of

$$ {}_4\Phi_3\left[\begin{matrix} a, & b, & c, & d; \\ & aq/b, & aq/c, & aq/d \end{matrix} q^2\right], \tag{4} $$

where $bcd = a\sqrt{q}$, and a is restricted as before. It is now evident that Jackson's first theorem gives the others immediately, since (2) is derived from (3) by substituting for e and letting $d \to 0$, while (4) is obtained from (3) by merely taking $e = \sqrt{(aq)}$.

A similar method could be used to find formulae giving transformations of well-poised series which cannot be summed, the principal parameter a having the special form of this paper. For example, from the transformation† of a terminating well-poised series ${}_8\Phi_7$ into a Saalschützian ${}_4\Phi_3$, we can find a transformation for the series

$$ {}_5\Phi_4\left[\begin{matrix} a, & b, & c, & d, & e; \\ & aq/b, & aq/c, & aq/d, & aq/e \end{matrix} a^{\frac{3}{2}}q^3/bcde\right], $$

* F. H. Jackson, *Messenger of Math.* 50 (1921), 109 (14).

† See my tract, already quoted, § 8.5 (2).

where a has the special form. The details of these results seem hardly to be worth working out, but I think it should be noted that there exist such straightforward analogues to results for ordinary hypergeometric series when the basic series terminate because of the special form of the principal parameter a. There seems to be no reason to suppose that such straightforward analogues exist when the series do not terminate, or when they terminate because of special forms of the other numerator parameters.

In conclusion I note explicitly the formula

$$\sum_{r=0}^{2N} (-1)^r \binom{2N}{r}^3 q^{\frac{1}{2}\{3(N-r)^2+(N-r)\}} = (-1)^N \frac{(q)_{3N}}{\{(q)_N\}^3},$$

which generalizes the formula for the sum of the cubes of the coefficients in the binomial expansion of $(1-x)^{2N}$. It is, of course, a particular case of Jackson's Theorem 2.

APPROXIMATIVE RIEMANN-SUMS FOR IMPROPER INTEGRALS

By D. R. DICKINSON (*Durham*)

[Received 21 June 1941]

1. LET $f(x)$ be a real function, defined in the finite interval $a < x \leqslant b$. We denote by D a dissection

$$a = x_0 < \xi_0 \leqslant x_1 \leqslant \xi_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \xi_n \leqslant x_{n+1} = b,$$

and we write
$$F(D) = \sum_{r=0}^{n} f(\xi_r)(x_{r+1} - x_r),$$

with a similar notation for other functions.

The points $\{x_r\}$ we call the 'points of subdivision' of D, and the points $\{\xi_r\}$ the 'representative points', and we write

$$\eta(D) = \max_{0 \leqslant r \leqslant n} (x_{r+1} - x_r).$$

Now, if the function $f(x)$ is Riemann-integrable in the interval (a, b), we know that

$$F(D) \to \int_a^b f(x)\, dx$$

as $\eta(D)$ tends to zero. In this paper I investigate the behaviour of $F(D)$ when the integral exists, not as a Riemann integral, but as the Cauchy integral

$$\int_a^b f(x)\, dx = \lim_{\delta \to +0} \int_{a+\delta}^b f(x)\, dx.$$

We suppose, then, that the function $f(x)$ is Riemann-integrable in the interval $(a+\delta, b)$, for $0 < \delta < b-a$, and that, as δ tends to zero through positive values, the integral

$$\int_{a+\delta}^b f(x)\, dx$$

tends to a finite limit, which we denote by

$$I = \int_a^b f(x)\, dx.$$

To avoid repetition we call this condition C.

Without some further restriction both on the function $f(x)$ *and* on

the dissection D it is no longer true that $F(D)$ tends to the limit I as $\eta(D)$ tends to zero. In §§ 2, 3 we shall consider some sets of conditions sufficient to ensure that this is the case.

2. The general problem with which we are dealing involves a complicated 'double limit'. For the present we shall confine ourselves to the case where the function $f(x)$ is monotonic in the interval (a, b).

Suppose, then, that $f(x)$ is a monotonic decreasing function in the interval (a, b), that $f(x) \to \infty$ as $x \to a$ from above,† and that $f(x)$ satisfies condition C of § 1.

It is still not true that $F(D) \to I$ as $\eta(D) \to 0$, unless some restriction is imposed upon the dissection D. I give here two separate restrictions on D, sufficient to ensure that $F(D) \to I$ as $\eta(D) \to 0$. The first restriction is that the first sub-interval (x_0, x_1) shall not be too small compared with the maximum sub-interval of D, and that the representative point ξ_0 shall not be chosen too near to the point x_0. The second restriction is that, in all the sub-intervals of D, the representative points shall not be chosen too near to the left-hand ends of their respective sub-intervals.

We first require the following lemma.

Lemma. *If $f(x)$ obeys the conditions stated above, then*

$$(x-a)f(x) \to 0 \quad \textit{as} \quad x \to a+0.$$

Let ϵ be any given positive number. Since $f(x)$ is Cauchy-integrable, we can choose a positive number δ such that, if $a < x < a+\delta$, then

$$\int_a^x f(t)\, dt < \epsilon.$$

Since $f(x)$ is a decreasing function, we then have

$$(x-a)f(x) < \epsilon.$$

This proves the lemma, since $f(x)$ is positive for values of x sufficiently near to a.

Theorem 1. *Let $0 < \lambda \leqslant 1$, $0 < \mu \leqslant 1$. Then, with the above restrictions on the function $f(x)$, if we confine ourselves to dissections D for which*

$$\text{(i)} \quad x_1 - a \geqslant \lambda\eta(D),$$

and

$$\text{(ii)} \quad \xi_0 - x_0 \geqslant \mu(x_1 - x_0),$$

it is true that $F(D) \to I$ as $\eta(D) \to 0$.

† Otherwise $f(x)$ is Riemann-integrable in (a, b).

Write

$$s(D) = \sum_{r=1}^{n} f(x_{r+1})(x_{r+1}-x_r),$$

$$S(D) = \sum_{r=1}^{n} f(x_r)(x_{r+1}-x_r).$$

Since $f(x)$ is a decreasing function,

$$s(D) \leqslant F(D) \leqslant S(D)+f(\xi_0)(x_1-x_0).$$

Now $f(\xi_0)$ is positive if $\eta(D)$ is small enough, and then, by condition (ii),

$$f(\xi_0)(x_1-x_0) \leqslant \frac{1}{\mu} f(\xi_0)(\xi_0-x_0).$$

It follows from the lemma that

$$f(\xi_0)(x_1-x_0) \to 0$$

as $\eta(D) \to 0$. It will, therefore, be sufficient to prove that, as $\eta(D) \to 0$, the sums $s(D)$ and $S(D)$ both tend to the limit I.

We first show that, as $\eta(D) \to 0$,

$$S(D)-s(D) \to 0. \tag{1}$$

We have, since $f(x)$ is a decreasing function,

$$S(D)-s(D) = \sum_{r=1}^{n} \{f(x_r)-f(x_{r+1})\}(x_{r+1}-x_r)$$

$$\leqslant \eta(D) \sum_{r=1}^{n} \{f(x_r)-f(x_{r+1})\} = \eta(D)\{f(x_1)-f(b)\}.$$

Certainly $\eta(D)f(b)$ tends to zero. Furthermore, $f(x_1)$ is positive if $\eta(D)$ is small enough, and then, by condition (i),

$$0 < \eta(D)f(x_1) \leqslant \frac{1}{\lambda}(x_1-a)f(x_1).$$

It follows from the lemma that $\eta(D)f(x_1)$ also tends to zero. This proves (1); hence, if ϵ is any given positive number, we can choose a positive number η_0 such that

$$S(D) < s(D)+\tfrac{1}{2}\epsilon \tag{2}$$

for all dissections D, of the type under consideration, for which $\eta(D) < \eta_0$. We can, moreover, choose η_0 so small that, if $x_1-a < \eta_0$,

$$I-\tfrac{1}{2}\epsilon \leqslant \int_{x_1}^{b} f(x)\,dx \leqslant I. \tag{3}$$

We have, also,

$$s(D) \leqslant \int_{x_1}^{b} f(x)\,dx \leqslant S(D). \tag{4}$$

Hence, combining the inequalities (2), (3), (4), we have, for $\eta(D) < \eta_0$,

$$I-\epsilon < s(D) \leqslant I.$$

This proves that $s(D) \to I$ as $\eta(D) \to 0$. Since $S(D)-s(D) \to 0$, it follows immediately that $S(D) \to I$ as $\eta(D) \to 0$. This completes the proof.

THEOREM 2. *Let $0 < \mu \leqslant 1$. Then, with the same restrictions as before on the function $f(x)$, if we confine ourselves to dissections D for which*

$$\xi_r - x_r \geqslant \mu(x_{r+1}-x_r) \quad (r = 0, 1,..., n),$$

it is true that $F(D) \to I$ as $\eta(D) \to 0$.

By Theorem 1, with $\lambda = \frac{1}{2}$, if ϵ is any given positive number, we can choose η_0 so small that, if $\eta(D) < \eta_0$, then

$$|F(D)-f(\xi_0)(x_1-x_0)-I| < \tfrac{1}{2}\epsilon, \tag{5}$$

provided that $x_1 - a \geqslant \frac{1}{2}\eta(D)$.

Now consider any dissection D, of the type under consideration, such that $\eta(D) < \eta_0$. Let x_{m+1} be the first point of subdivision of D such that

$$x_{m+1} - a \geqslant \tfrac{1}{2}\eta(D),$$

and write

$$F(D) = F(D')+F(D''),$$

where $F(D') = \sum_{r=0}^{m} f(\xi_r)(x_{r+1}-x_r), \qquad F(D'') = \sum_{r=m+1}^{n} f(\xi_r)(x_{r+1}-x_r).$

We have immediately, by (5),

$$|F(D'')-I| < \tfrac{1}{2}\epsilon.$$

It remains to show that $F(D')$ is small.

Now, since $f(x)$ is a decreasing function,

$$\int_{x_r}^{x_{r+1}} f(x)\,dx \geqslant f(\xi_r)(\xi_r - x_r) \geqslant \mu f(\xi_r)(x_{r+1}-x_r),$$

and thus

$$f(\xi_r)(x_{r+1}-x_r) \leqslant \frac{1}{\mu}\int_{x_r}^{x_{r+1}} f(x)\,dx.$$

Hence

$$F(D') \leqslant \frac{1}{\mu}\int_{x_0}^{x_m} f(x)\,dx < \tfrac{1}{2}\epsilon$$

for sufficiently small values of $\eta(D)$. Choosing $\eta(D)$ small enough for $F(D')$ to be positive, we then have

$$|F(D)-I| \leqslant |F(D'')-I|+F(D') < \epsilon.$$

This proves the theorem.

As an example consider the beta function

$$\mathrm{B}(1-\alpha, 1-\beta) = \int_0^1 \frac{dx}{x^\alpha(1-x)^\beta},$$

where $\alpha > 0$, $\beta > 0$. We show that

$$\mathrm{B}(1-\alpha, 1-\beta) = \lim_{n\to\infty} n^{\alpha+\beta-1} \sum_{r=1}^{n-1} \frac{1}{r^\alpha(n-r)^\beta}. \tag{6}$$

This result is an immediate consequence of Theorem 1 in the case $\beta = 0$. Similarly (6) is true for $\alpha = 0$, and the general case is then easily deduced from the fact that

$$\frac{1}{x^\alpha(1-x)^\beta} - \frac{1}{x^\alpha} - \frac{1}{(1-x)^\beta}$$

is bounded and Riemann-integrable in the interval $(0, 1)$.†

3. We shall now extend the results of Theorems 1 and 2 to a wider class of functions.

With the same restrictions on the types of allowable dissections, it is still true that $F(D) \to I$ as $\eta(D) \to 0$ if we replace the restriction that $f(x)$ should be monotonic by the condition

$$|f(x)| \leqslant f_1(x),$$

where $f_1(x)$ is some monotonic function satisfying condition C of §1.

This result is included in the following theorem.

THEOREM 3. *Suppose that*

(i) *the function $f(x)$ satisfies condition* C,

(ii) $|f(x)| \leqslant f_1(x)$ *for* $a < x \leqslant b$,

(iii) $f_1(x)$ *is a monotonic decreasing function in this interval, and also satisfies condition* C,

(iv) Δ *is a class of dissections (containing dissections D for which $\eta(D)$ is arbitrarily small) such that*

$$F_1(D) \to I_1 = \int_a^b f_1(x)\,dx$$

as $\eta(D) \to 0$, provided we confine ourselves to dissections which belong to the class Δ.

† The results for $\alpha = 0$, $\beta = 0$ can, of course, easily be verified directly without appeal to Theorem 1.

Under these conditions it is true, also, that

$$F(D) \to I = \int_a^b f(x)\,dx$$

as $\eta(D) \to 0$, provided again that we only consider dissections belonging to the class Δ.

Let ϵ be any given positive number. We can choose a positive number δ such that

$$\int_a^{a+\delta} f_1(x)\,dx < \tfrac{1}{6}\epsilon, \tag{7}$$

and therefore

$$\left|\int_a^{a+\delta} f(x)\,dx\right| < \tfrac{1}{6}\epsilon. \tag{8}$$

We write

$$I' = \int_a^{a+\delta} f(x)\,dx, \qquad I'' = \int_{a+\delta}^{b} f(x)\,dx,$$

with similar definitions for I_1' and I_1''.

Now I'' and I_1'' are ordinary Riemann integrals. We can, therefore, determine η_0 such that, if D'' is any dissection of the interval $(a+\delta, b)$ for which $\eta(D'') < \eta_0$, then

$$|F(D'')-I''| < \tfrac{1}{6}\epsilon \tag{9}$$

and

$$|F_1(D'')-I_1''| < \tfrac{1}{6}\epsilon. \tag{10}$$

Let D be any dissection, belonging to the class Δ, such that $\eta(D) < \eta_0$. If D does not have the point $a+\delta$ as a point of subdivision, we add this point to it and call the modified dissection D^*; otherwise D^* simply denotes D. Since the difference between $F_1(D^*)$ and $F_1(D)$ tends to zero with $\eta(D)$, we can choose $\eta_1 \leqslant \eta_0$ such that

$$|F_1(D^*)-I_1| < \tfrac{1}{6}\epsilon \quad (\eta(D) < \eta_1). \tag{11}$$

Now the dissection D gives rise to a dissection D' of the interval $(a, a+\delta)$ and a dissection D'' of the interval $(a+\delta, b)$. We have, moreover,

$$\begin{aligned} |F(D')| &\leqslant F_1(D') = F_1(D^*)-I_1+I_1'-\{F_1(D'')-I_1''\} \\ &\leqslant |F_1(D^*)-I_1|+I_1'+|F_1(D'')-I_1''| < \tfrac{1}{2}\epsilon \end{aligned} \tag{12}$$

by (11), (7), (10). Hence

$$\begin{aligned} |F(D^*)-I| &= |F(D')-I'+F(D'')-I''| \\ &\leqslant |F(D')|+|I'|+|F(D'')-I''| < \tfrac{5}{6}\epsilon \end{aligned}$$

by (12), (8), (9). Since the difference between $F(D^*)$ and $F(D)$ tends to zero with $\eta(D)$ we have, for sufficiently small values of $\eta(D)$,

$$|F(D)-I| < \epsilon.$$

This proves the theorem.

4. The following example will perhaps clarify the position. We construct a function $\phi(x)$ for which it is no longer true that

$$\Phi(D) \to \int_a^b \phi(x)\,dx$$

as $\eta(D) \to 0$, even though we restrict ourselves to dissections in which each sub-interval is of the same length, and take the representative points at the mid-points of the sub-intervals.

Let r be a positive integer or zero. We define our function in the interval $(2^{-(r+1)}, 2^{-r})$ as follows. In the first place we write

$$\phi(x) = 0 \quad \text{for} \quad 2^{-(r+1)} \leqslant x \leqslant 3.2^{-(r+2)}-2^{-(2r+3)}$$

and for $$3.2^{-(r+2)}+2^{-(2r+3)} \leqslant x \leqslant 2^{-r}.$$

We then take $$\phi(3.2^{-(r+2)}) = 2^r,$$

and $\phi(x)$ we now define in the remainder of the interval by linear interpolation. This process being repeated for each value of r, our function is defined throughout the interval $0 < x \leqslant 1$.

In the interval $(2^{-(r+1)}, 2^{-r})$ the graph of $\phi(x)$ consists of a triangle of height 2^r and base $2^{-(2r+2)}$, imposed centrally above the interval. The area of this triangle is $2^{-(r+3)}$. It follows that $\phi(x)$ is Cauchy-integrable in the interval $(0, 1)$, and that

$$\int_0^1 \phi(x)\,dx = \sum_{r=0}^{\infty} \frac{1}{2^{r+3}} = \frac{1}{4}.$$

Now denote by D_n the dissection obtained by dividing the interval $(0, 1)$ into 2^{n+1} equal sub-intervals, and taking the representative points at the mid-points of the sub-intervals. Consider $\Phi(D_n)$. The contribution to this sum arising from the sub-interval $(2^{-(n+1)}, 2^{-n})$ is

$$2^n \times 2^{-(n+1)} = \tfrac{1}{2},$$

and hence $$\varliminf_{n\to\infty} \Phi(D_n) \geqslant \tfrac{1}{2} > \int_0^1 \phi(x)\,dx.$$

In fact, as is easily verified, $\Phi(D_n) \to \frac{3}{4}$ as $n \to \infty$.

5. Integrals over an infinite range

For these integrals the following result is true. The proof is left to the reader.

THEOREM 4. *Suppose that*

(i) *for $X > a$, $f(x)$ is Riemann-integrable in the interval (a, X);*

(ii) $|f(x)| \leqslant f_1(x)$ *for* $x \geqslant a$;

(iii) $f_1(x)$ *is a monotonic decreasing function for $x \geqslant a$, and the infinite integral*

$$\int_a^\infty f_1(x)\,dx$$

exists finitely;

(iv) *D is a dissection*

$$a = x_0 \leqslant \xi_0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant \xi_n \leqslant x_{n+1} \leqslant \ldots,$$

such that x_n tends to infinity with n and $x_{n+1}-x_n$ is bounded.

Under these conditions the infinite sum

$$F(D) = \sum_{r=0}^{\infty} f(\xi_r)(x_{r+1}-x_r)$$

exists, and

$$F(D) \to \int_a^\infty f(x)\,dx$$

as $\eta(D) \to 0$, where

$$\eta(D) = \overline{\text{bound}}(x_{n+1}-x_n).$$

As an example consider the function $x^{-(1+\alpha)}$. For $\alpha > 0$,

$$\int_1^\infty \frac{dx}{x^{1+\alpha}} = \frac{1}{\alpha}.$$

Let D be a dissection of $(1, \infty)$ in which each sub-interval is of the same length δ, and take the representative points at the right-hand ends of the sub-intervals. Then

$$F(D) = \delta \sum_{r=1}^{\infty} \frac{1}{(1+r\delta)^{1+\alpha}} \to \frac{1}{\alpha}$$

as $\delta \to +0$, if $\alpha > 0$. Writing $\delta = 1/n$, we deduce that

$$n^\alpha \sum_{r=n+1}^{\infty} \frac{1}{r^{1+\alpha}} \to \frac{1}{\alpha}$$

as $n \to \infty$.

ON TCHEBYCHEFF POLYNOMIALS (II)

By D. R. DICKINSON (*Durham*)

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1. We say that a system

$$\phi_0(x),\ \phi_1(x),\ \dots,\ \phi_n(x)$$

of $n+1$ real functions, continuous in a closed interval (a,b), forms a T-system of order n in (a,b) if no linear combination

$$P(x) = A_0\phi_0(x)+A_1\phi_1(x)+\dots+A_n\phi_n(x),$$

with real coefficients, has more than n distinct zeros in this interval unless the coefficients are all zero; and we then call $P(x)$ a T-polynomial (or simply a polynomial) of the system. We shall suppose throughout the first part of this paper that we are dealing with a definite T-system of order n, where $n > 2$.

In a previous paper* I have given a discussion on double zeros of T-polynomials. In the present work I go on to consider triple zeros. I give definitions of two kinds of triple zeros, namely *weak* triple zeros and *strong* triple zeros, and then proceed to show that we can always construct a polynomial of our system having a weak triple zero at any given point, but that the same is not true for strong triple zeros. It will be seen that it is the stronger definition which corresponds more closely to the analytical case, and it is in the possible non-existence of strong zeros that the general T-system differs fundamentally from an analytical T-system.

2. For convenience of reference we first recall the definitions of simple and double zeros; we now differentiate between weak and strong double zeros.

Let $\bar{x}$ be a zero of $P(x)$, a T-polynomial of our system. We say that $\bar{x}$ is a *simple zero* (at least) of $P(x)$ if the polynomial $P(x)$ changes sign as x passes through $\bar{x}$. If $P(x)$ preserves the same sign as x passes through $\bar{x}$, then we say that $\bar{x}$ is a *weak double zero* (at least) of $P(x)$; if, in addition,

$$P(x)/Q(x) \to 0 \quad \text{as} \quad x \to \bar{x},$$

for any polynomial $Q(x)$ having a simple zero at $\bar{x}$ and at least $n-2$ other zeros, then we call $\bar{x}$ a *strong double zero* of $P(x)$.

* D. R. Dickinson, 'On Tchebycheff Polynomials': *Quart. J. of Math.* (Oxford), 10 (1939), 277–82.

The main results in connexion with double zeros are as follows:

THEOREM 1. *If s is the number of simple zeros and d the number of double zeros (weak or strong) of a T-polynomial $P(x)$ of our system, then*

$$s+2d \leqslant n.$$

THEOREM 2. *If $s+2d = n$, and we are given $s+d$ distinct points $x_1, x_2,\ldots, x_s; y_1, y_2,\ldots, y_d$ such that*

$$a \leqslant x_i \leqslant b \quad (i = 1, 2,\ldots, s), \qquad a < y_j < b \quad (j = 1, 2,\ldots, d),$$

then we can construct a T-polynomial of our system having simple zeros at the points $x_1, x_2,\ldots, x_s$ and weak double zeros at the points $y_1, y_2,\ldots, y_d$.

In the analytical case the solution to Theorem 2 is unique, except for a constant multiplier. I have, however, shown by an example that this is not true for T-systems in general. This example* also shows that we cannot always construct a polynomial of our system having a *strong* double zero at any given point.

3. We now proceed to the definitions of triple zeros. Let $P(x)$ be a polynomial of our system having a zero at $\bar{x}$, an interior point of the interval (a, b), and changing sign as x passes through $\bar{x}$. Suppose, further, that for any polynomial $Q(x)$ having a simple zero at $\bar{x}$ and at least $n-2$ other zeros (double zeros being counted twice),

$$P(x)/Q(x) \to 0 \quad \text{as} \quad x \to \bar{x}. \tag{1}$$

In these circumstances we call $\bar{x}$ a *weak triple zero* of $P(x)$. If, in addition, (1) is still true when $Q(x)$ is replaced by any polynomial having a double zero at $\bar{x}$ and at least $n-3$ other zeros, then we call $\bar{x}$ a *strong triple zero* of $P(x)$.

Definitions similar to the above can easily be framed for zeros of higher orders, and the results established below may be extended to cover such zeros. As, however, the proofs become long and troublesome, without entailing any essentially new ideas, we content ourselves with a study of triple zeros only.

THEOREM 3. *If s is the number of simple zeros, d the number of double zeros (weak or strong), and t the number of triple zeros (weak or strong) of a T-polynomial $P(x)$ of our system, then*

$$s+2d+3t \leqslant n.$$

This theorem is proved by a slight extension of the argument used

* See § 6 below.

to establish Theorem 1. As we shall be using a similar argument in §4 below, I omit details of the proof.

4. Before going on to consider the problem of constructing T-polynomials having previously assigned weak triple zeros, we shall first establish three lemmas.

In what follows, unless anything is said to the contrary, we suppose that any polynomial which occurs in the argument may have simple, double, and triple zeros. In counting the zeros of a polynomial we count its double zeros twice and its triple zeros three times.

LEMMA 1. *If $P(x)$, $Q(x)$ are any two T-polynomials of our system, both having a zero at $\bar{x}$, an interior point of (a,b), then*

$$\lim_{x\to\bar{x}-0}\frac{Q(x)}{P(x)} \quad \textit{and} \quad \lim_{x\to\bar{x}+0}\frac{Q(x)}{P(x)}$$

both exist (finitely or infinitely).

Take any real number λ, and consider the polynomial $Q(x)-\lambda P(x)$. This polynomial has at most n zeros, and hence the ratio $Q(x)/P(x)$ cannot assume the value λ at more than n distinct points of (a,b). Since $Q(x)/P(x)$ is continuous at all points near $\bar{x}$, Lemma 1 follows at once.

We have assumed here that $\bar{x}$ is an interior point of (a,b). If $\bar{x}$ is an end point, then evidently the appropriate one-sided limit of $Q(x)/P(x)$ exists.

LEMMA 2. *Suppose that $P(x)$, $Q(x)$ are two T-polynomials of our system, both having a zero at $\bar{x}$, an interior point of (a,b). Suppose, in addition, that $Q(x)$ changes sign as x passes through $\bar{x}$, and has at least $n-2$ zeros other than $\bar{x}$. In these circumstances,*

$$\lim_{x\to\bar{x}-0}\left|\frac{Q(x)}{P(x)}\right| > 0 \quad \textit{and} \quad \lim_{x\to\bar{x}+0}\left|\frac{Q(x)}{P(x)}\right| > 0.$$

It will be sufficient to prove the first of these inequalities. In the first place we shall consider the case in which $Q(x)$ has exactly n distinct zeros (necessarily simple). We may evidently suppose that $P(x)$ and $Q(x)$ have the same sign to the immediate left of the point $\bar{x}$. Now let $x_1, x_2,\dots, x_m$ be the zeros of $Q(x)$, other than a or b, which are not zeros of $P(x)$, and for convenience of notation write $\bar{x} = x_{m+1}$.

We suppose, to begin with, that a and b are not *both* zeros of $Q(x)$. Denote by $2\delta_r$ ($r = 1, 2,..., m+1$) the distance of x_r from the nearest other zero of $Q(x)$ or $P(x)$, and write

$$\begin{aligned} \delta &= \min(\delta_r) && (r = 1, 2,..., m+1), \\ N_r^+ &= \overline{\text{bound}}\{|Q(x)|\} && (x_r \leqslant x \leqslant x_r+\delta), \\ N_r^- &= \overline{\text{bound}}\{|Q(x)|\} && (x_r-\delta \leqslant x \leqslant x_r), \\ N &= \min(N_r^+, N_r^-) && (r = 1, 2,..., m+1), \\ M &= \overline{\text{bound}}\{|P(x)|\} && (a \leqslant x \leqslant b). \end{aligned}$$

Choose a positive number λ such that $\lambda M < N$. Then to any point x_r ($r = 1, 2,..., m$) there corresponds at least one zero of the polynomial $R(x) = Q(x)-\lambda P(x)$ in the open interval $(x_r-\delta, x_r+\delta)$. The common zeros of $P(x)$ and $Q(x)$ are also zeros of $R(x)$ and so* we have accounted for at least $n-1$ distinct zeros of $R(x)$.

Suppose now, if possible, that

$$\lim_{x\to\bar{x}-0} \frac{Q(x)}{P(x)} = 0. \tag{1}$$

The polynomial $R(x)$ will then have at least one additional zero in the interval $(\bar{x}-\delta, \bar{x})$ and either a double zero at $\bar{x}$ or at least one additional zero in the interval $(\bar{x}, \bar{x}+\delta)$. Thus $R(x)$ will have at least $n+1$ zeros, which is impossible by Theorem 3. Hence (1) cannot be true.

If a and b are both zeros of $Q(x)$, we proceed as follows. Let x_1 be the zero (other than a) of $Q(x)$ nearest to a, and choose x' such that $a < x' < x_1$. We can construct a polynomial $Q'(x)$ having simple zeros at the zeros of $Q(x)$ other than a, and also at x'. We deduce, by reasoning similar to that used above, that

$$\lim_{x\to\bar{x}-0} \left|\frac{Q(x)}{Q'(x)}\right| > 0.$$

But, by what we have already proved,

$$\lim_{x\to\bar{x}-0} \left|\frac{Q'(x)}{P(x)}\right| > 0,$$

and so
$$\lim_{x\to\bar{x}-0} \left|\frac{Q(x)}{P(x)}\right| > 0$$

in this case also.

* If neither a nor b is a zero of $Q(x)$, we have accounted for all the n zeros of $R(x)$.

Considering now the general case, let $x_1, x_2, \ldots, x_s$ be the simple zeros of $Q(x)$; $y_1, y_2, \ldots, y_d$ its double zeros; and $z_1, z_2, \ldots, z_t$ its triple zeros. Denote by 2η the least distance between two distinct zeros.

We construct a polynomial $Q_1(x)$, having exactly n simple zeros, as follows. In the first place we make $x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_d$ zeros of $Q_1(x)$. Consider now a typical triple zero z_k of $Q(x)$. If there is an even number of double zeros of $Q(x)$ between z_k and $\bar{x}$, we simply make z_k a zero of $Q_1(x)$. If, however, there is an odd number of double zeros of $Q(x)$ between z_k and $\bar{x}$, then we take as zeros of $Q_1(x)$ both z_k and whichever of the points $z_k+\eta$, $z_k-\eta$ lies nearer to $\bar{x}$. We make up the number of zeros of $Q_1(x)$ to n by giving this polynomial the appropriate further number of simple zeros to the left of all the double and triple zeros of $Q(x)$.

By reasoning as in the first part of the proof, we now deduce that

$$\lim_{x\to\bar{x}-0}\left|\frac{Q(x)}{Q_1(x)}\right| > 0.$$

But, by what we have already proved,

$$\lim_{x\to\bar{x}-0}\left|\frac{Q_1(x)}{P(x)}\right| > 0.$$

Hence

$$\lim_{x\to\bar{x}-0}\left|\frac{Q(x)}{P(x)}\right| > 0$$

in the general case also.

Now let $P(x)$ be a polynomial of our system having a weak triple zero at $\bar{x}$, and let $Q(x)$ be any other polynomial having a simple zero at $\bar{x}$ and at least $n-2$ other zeros.* We deduce at once from Lemma 2 that

$$P(x)/Q(x) \to 0 \quad \text{as} \quad x \to \bar{x}.$$

We have, moreover, the following result, which we shall require in the sequel.

LEMMA 3. *Let $P(x)$ be a T-polynomial of our system which has a zero at $\bar{x}$ and changes sign as x passes through $\bar{x}$. If we can find* ONE *polynomial $Q_0(x)$, having a zero at $\bar{x}$ and such that*

$$P(x)/Q_0(x) \to 0 \quad \textit{as} \quad x \to \bar{x},$$

then $P(x)$ has a weak triple zero at $\bar{x}$.

* The polynomial $Q(x)$ is now, of course, allowed to have triple zeros: otherwise there is nothing to prove.

5. We now come to the main result in connexion with weak triple zeros of T-polynomials.

THEOREM 4. *If $s+2d+3t = n$, and we are given $s+d+t$ distinct points $x_1, x_2,..., x_s$; $y_1, y_2,..., y_d$; $z_1, z_2,..., z_t$ such that*

$$a \leqslant x_i \leqslant b \quad (i = 1, 2,..., s),$$
$$a < y_j < b \quad (j = 1, 2,..., d),$$
$$a < z_k < b \quad (k = 1, 2,..., t),$$

then we can construct a T-polynomial of our system having simple zeros at the points $x_1, x_2,..., x_s$, weak double zeros at the points $y_1, y_2,..., y_d$, and weak triple zeros at the points $z_1, z_2,..., z_t$.

The proof is by induction on t. The theorem is true for $t = 0$, since it then reduces to Theorem 2; suppose it true for $t = t_1-1$, and consider the case $t = t_1$, writing t for t_1.

Take four points p, q, α, β such that

$$p < \alpha < z_t < \beta < q$$

and, further, such that the interval (p, q) contains none other of the given points. By hypothesis we can construct four polynomials

$$P_\alpha(x),\ Q_\alpha(x),\ P_\beta(x),\ Q_\beta(x)$$

of our system, with the following properties. In the first place, each of the four polynomials has simple zeros at $z_t, x_1, x_2,..., x_s$, weak double zeros at $y_1, y_2,..., y_d$, and weak triple zeros at $z_1, z_2,..., z_{t-1}$. In addition we make

$$\begin{aligned} P_\alpha(\alpha) &= P_\alpha(p) = 0, & P_\alpha(q) &= 1,\\ Q_\alpha(\alpha) &= Q_\alpha(q) = 0, & Q_\alpha(p) &= 1,\\ P_\beta(\beta) &= P_\beta(p) = 0, & P_\beta(q) &= 1,\\ Q_\beta(\beta) &= Q_\beta(q) = 0, & Q_\beta(p) &= 1, \end{aligned}$$

the zeros being necessarily simple, by Theorem 3.

Now let $\qquad \alpha < z < z_t \quad \text{or} \quad z_t < z \leqslant \beta,$

and write

$$\lambda(z) = -\frac{P_\beta(z)}{P_\alpha(z)}, \qquad \mu(z) = -\frac{Q_\beta(z)}{Q_\alpha(z)}, \qquad \nu(z) = +\frac{Q_\alpha(z)}{P_\alpha(z)}.$$

Write, further,

$$\begin{aligned} \underline{\lambda} &= \lim_{z\to z_t+0} \lambda(z), & \bar{\lambda} &= \lim_{z\to z_t-0} \lambda(z),\\ \underline{\mu} &= \lim_{z\to z_t+0} \mu(z), & \bar{\mu} &= \lim_{z\to z_t-0} \mu(z),\\ \underline{\nu} &= \lim_{z\to z_t+0} \nu(z), & \bar{\nu} &= \lim_{z\to z_t-0} \nu(z). \end{aligned}$$

These limits are, by Lemma 2, finite and non-zero. Furthermore, the polynomials $P_\alpha(x)$ and $P_\beta(x)$ are of different sign in the interval (α, β) and have the same sign outside this interval. It follows that $\lambda(z)$ is non-negative in the ranges of z considered, and that the polynomial $P_\beta(x)+\lambda(z)P_\alpha(x)$ takes the fixed value $P_\beta(\alpha)$ at the point α and has the required zeros and none other outside the interval (α, β). We have thus accounted for $s+2d+3t-3$ zeros of $P_\beta(x)+\lambda(z)P_\alpha(x)$. This polynomial has also zeros at the points p, z, and z_t. Hence these zeros must be simple and the polynomial can have no further zeros. Hence, since $\lambda(z)$ is continuous, it is monotone decreasing in the ranges of z considered and

$$0 < \underline{\lambda} \leqslant \bar{\lambda}.$$

Similarly, $\quad 0 < \underline{\mu} \leqslant \bar{\mu} \quad$ and* $\quad 0 < \underline{\nu} \leqslant \bar{\nu}$.

Now write

$$A_z(x) = P_\beta(x)+\lambda(z)P_\alpha(x),$$
$$B_z(x) = Q_\beta(x)+\mu(z)Q_\alpha(x),$$

and

$$\bar{A}(x) = P_\beta(x)+\bar{\lambda}P_\alpha(x),$$
$$\bar{B}(x) = Q_\beta(x)+\bar{\mu}Q_\alpha(x),$$

with similar definitions for $\underline{A}(x)$ and $\underline{B}(x)$. Write, further,

$$\underline{\sigma}(z) = \lim_{x\to z_t+0} \frac{B_z(x)}{A_z(x)} = \underline{\nu}\frac{\underline{\mu}-\mu(z)}{\underline{\lambda}-\lambda(z)},$$

and

$$\bar{\sigma}(z) = \lim_{x\to z_t-0} \frac{B_z(x)}{A_z(x)} = \bar{\nu}\frac{\bar{\mu}-\mu(z)}{\bar{\lambda}-\lambda(z)}.$$

Reasoning as before, we see that

$$0 < \underline{\sigma}(z) \leqslant \bar{\sigma}(z) < \frac{Q_\beta(\alpha)}{P_\beta(\alpha)}.$$

Moreover, the polynomial

$$\bar{C}_z(x) = B_z(x)-\underline{\sigma}(z)A_z(x)$$

will have the required zeros outside the interval (p, q), a simple zero at the point z, and a double zero at the point z_t.

Now, observing that

$$\underline{\sigma}(z) = \frac{\underline{\nu}}{\nu(z)}\,\frac{\underline{B}(z)}{\underline{A}(z)},$$

* The polynomials $P_\alpha(z)$ and $Q_\alpha(z)$ have the same sign throughout the interval (p, q) and are of different sign outside this interval.

and that $\sigma(z)$ is bounded,* we deduce that

$$\underline{\sigma} = \lim_{z\to z_l-0} \underline{\sigma}(z)$$

exists finitely.

We can now show that the polynomial

$$\begin{aligned}\bar{C}(x) &= \bar{B}(x) - \underline{\sigma}\bar{A}(x)\\ &= Q_\beta(x) + \bar{\mu}Q_\alpha(x) - \underline{\sigma}\{P_\beta(x) + \bar{\lambda}P_\alpha(x)\}\end{aligned}$$

has the properties we require. In the first place, in virtue of the signs of the coefficients, this polynomial has the required zeros outside the interval (p,q). Secondly,

$$\bar{C}_z(x) \to \bar{C}(x) \quad \text{as} \quad z \to z_l - 0,$$

from which we deduce that $\bar{C}(x)$ changes sign as x passes through z_l. Finally, we have

$$\frac{\bar{C}(x)}{P_\alpha(x)} = \nu(x)\{\bar{\mu} - \mu(x)\} - \underline{\sigma}\{\bar{\lambda} - \lambda(x)\}.$$

Evidently
$$\lim_{x\to z_l-0} \frac{\bar{C}(x)}{P_\alpha(x)} = 0. \tag{1}$$

Further,

$$\lim_{x\to z_l+0} \frac{\bar{C}(x)}{P_\alpha(x)} = \underline{\nu}\left\{(\bar{\mu} - \underline{\mu}) - (\bar{\lambda} - \underline{\lambda}) \lim_{z\to z_l-0} \frac{\underline{\mu} - \mu(z)}{\underline{\lambda} - \lambda(z)}\right\}.$$

Considering separately the cases $\underline{\lambda} = \bar{\lambda}$ and $\underline{\lambda} < \bar{\lambda}$, we see that†

$$\lim_{x\to z_l+0} \frac{\bar{C}(x)}{P_\alpha(x)} = 0. \tag{2}$$

In virtue of (1), (2), and Lemma 3 it now follows that the polynomial $\bar{C}(x)$ has a weak triple zero at the point z_l. This completes the proof of Theorem 4.

Uniqueness of the solution. In the first place the construction of the four functions

$$P_\alpha(x),\ Q_\alpha(x),\ P_\beta(x),\ Q_\beta(x)$$

may not be unique; in this case it is easy to see that the solution to Theorem 4 is not unique. Furthermore, if we write

$$\bar{\sigma} = \lim_{z\to z_l+0} \bar{\sigma}(z),$$

* We cannot apply Lemma 2 to show that $\underline{\sigma}$ is finite, since $\underline{A}(z)$ and $\underline{B}(z)$ both have *double* zeros at z_l.

† If $\underline{\lambda} = \bar{\lambda}$, then, since σ is finite, $\underline{\mu} = \bar{\mu}$.

then, by the same argument as before, the polynomial

$$\underline{C}(x) = \underline{B}(x) - \bar{\sigma}\underline{A}(x)$$
$$= Q_\beta(x) + \underline{\mu} Q_\alpha(x) - \bar{\sigma}\{P_\beta(x) + \underline{\lambda} P_\alpha(x)\}$$

will be a solution which in general will not be a constant multiple of $\bar{C}(x)$. Now

$$\bar{C}(p) = 1 + \bar{\mu} \quad \text{and} \quad \underline{C}(p) = 1 + \underline{\mu}.$$

It follows that, if ξ and η are positive numbers such that

$$\xi + \eta = 1,$$

then the polynomial

$$C(x) = \frac{\xi}{1+\bar{\mu}} \bar{C}(x) + \frac{\eta}{1+\underline{\mu}} \underline{C}(x)$$

will be a solution to Theorem 4 such that $C(p) = 1$.

6. In conclusion we construct a T-system of order 3, no polynomial of which has a *strong* triple zero at a particular point.

I have shown in my first paper that the functions

$$1, \quad |x|, \quad \sin x \tag{1}$$

form a T-system of order 2 in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The reader will see that no polynomial of this system can have a strong double zero at the origin. Now, integrating the functions (1) in turn, we see that the functions

$$1, \quad x, \quad x^2 \operatorname{sgn} x, \quad \cos x \tag{2}$$

form a T-system of order 3 in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. If $\phi(x)$ is any polynomial of this system which has a zero at the origin, then either

$$\phi(x) \sim cx \tag{3}$$

as $x \to 0$, in which case $\phi(x)$ changes sign as x passes through the origin; or

$$\phi(x) \sim cx^2 \tag{4}$$

as $x \to 0$ from one side at least. Here c is some constant. We see immediately that $\phi(x)$ cannot have a strong triple zero at the origin. The reader will easily verify that, if $\phi(x)$ has a simple zero at the origin and at least one other zero in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then (3) is true; and that the polynomial

$$ax^2 \operatorname{sgn} x + b(1 - \cos x) \quad (b \neq 0)$$

will have a weak triple zero at the origin if either $b = \pm 2a$, or $b+2a$ and $b-2a$ are of opposite sign.